Graph Reconstruction by Permutations

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Abstract

Let G and H be graphs with a common vertex set V, such that $G-i \cong H-i$, for all $i \in V$. Let p_i be the permutation of $V-i$ that maps $G-i$ to $H-i$, and let q_i denote the permutation obtained from p_i by mapping i to i. It is shown that certain algebraic relations involving the edges of G and the permutations $q_i q_j^{-1}$ and $q_i q_k^{-1}$, where $i, j, k \in V$ are distinct vertices, often force G and H to be isomorphic.

1. Permutations and Partial Permutations

Let G and H be graphs with a common vertex set V, such that $G-i \cong H-i$, for all $i \in V$. Ulam's conjecture, also known as the graph reconstruction conjecture (see Bondy [1]), states that this condition implies that $G \cong H$, if $|V| \geq 3$. All graphs that we will work with are simple graphs, with the same vertex set V. We will view a graph G as a set of edges, where each edge is an unordered pair of distinct vertices. We write $\binom{V}{2}$ for the set of all unordered pairs of vertices. Thus, $G \subseteq \binom{V}{2}$. The complement of G is $\overline{G} = \binom{V}{2} - G$. An unordered pair of vertices is denoted by $[x, y]$. (The square brackets are convenient when it is necessary to construct sets of unordered pairs.) It will often be convenient to write $[x, y]_G = 1$ to mean that $[x, y] \in G$, and $[x, y]_G = 0$ to mean that $[x, y] \notin G$. Given a permutation q of V, and a vertex $x \in V$, the image of x under q is denoted x^q. Then $[x, y]^q = [x^q, y^q]$. We also write G^q for the graph obtained from G by permuting each edge of G by q. Let p_i be an isomorphism of $G-i$ with $H-i$, so that $(G-i)^{p_i} = H-i$. Then p_i is a *partial permutation* of the set V. It does not map the point i.

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1.1 Definition. A *partial permutation* p of V is an injective mapping from a proper subset $X \subset V$ to a proper subset $Y \subset V$. A *partial automorphism* of a graph G is a partial permutation p such that for all $[x, y] \in \binom{V}{2}$:

1) if $[x, y] \in G$ and $[x, y]^p$ is defined, then $[x, y]^p \in G$; and
2) if $[x, y] \notin G$ and $[x, y]^p$ is defined, then $[x, y]^p \notin G$;

The inverse of a partial permutation is also a partial permutation. Notice that partial permutations can be composed. If p_i and p_j are partial permutations, so is $p_i p_j^{-1}$. The result is another partial permutation. Permutations and partial permutations are composed from left to right, so that the product $p_i p_j^{-1}$ means "first p_i, then p_j^{-1}". Partial permutations were applied to the graph reconstruction problem in Kocay [3].

We say that the image $[x, y]^{p_i p_j^{-1}}$ *exists* if $p_i p_j^{-1}$ can be applied to both x and y. We are given the graphs G and H such that $(G - i)^{p_i} = H - i$, for all i. Notice that for all $i, j \in V$, $p_i p_j^{-1}$ is a partial automorphism of G, and $p_j^{-1} p_i$ is a partial automorphism of H, because p_i and p_j are isomorphisms. Notice that if $[x, y]^{p_i p_j^{-1}}$ exists, then $[x, y] \in G$ if and only if $[x, y]^{p_i p_j^{-1}} \in G$, since $p_i p_j^{-1}$ is a partial automorphism of G.

1.2 Lemma. *Let $[x, y]$ be a pair of vertices of V. Then the image $[x, y]^{p_i p_j^{-1}}$ exists if and only if $x, y \notin \{i, j^{p_i^{-1}}\}$. The image $[x, y]^{p_j^{-1} p_i}$ exists if and only if $x, y \notin \{j, i^{p_j}\}$.*

The partial permutation p_i can be converted to a permutation q_i, by extending p_i to act on i, defining $i^{q_i} = i$. Notice that $q_i q_j^{-1}$ is in general not an automorphism of G.

1.3 Lemma. *There are two possibilities for the cycle structure of $q_i q_j^{-1}$. Either i and j are in the same cycle, or they are in different cycles.*

The two cases for the cycle structure of $q_i q_j^{-1}$ are illustrated in Figure 1. We say that $q_i q_j^{-1}$ is of type 1 if i and j are in the same cycle, and of type 2 if they are in different cycles. The arrow in the diagrams indicates the direction of the cycle, and also indicates those links in the cycle which $p_i p_j^{-1}$ cannot follow.

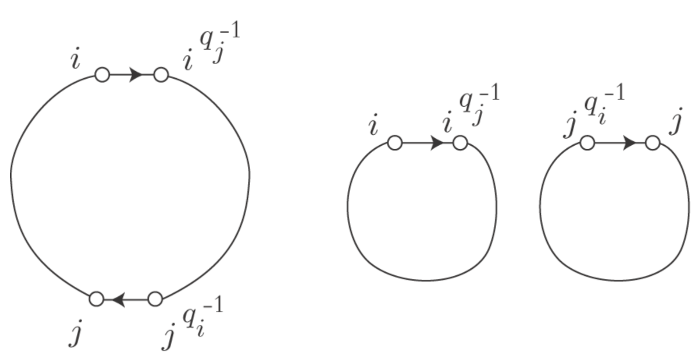

Figure 1, The cycle structure of $q_i q_j^{-1}$, types 1 and 2

We assume the basic theory of graph reconstruction. The reader is referred to the survey papers by Bondy and Hemminger [1], Bondy [2], and Nash-Williams [4]. In particular, given a fixed graph K with fewer than $|V|$ vertices, Kelly's lemma says that the number of subgraphs of G isomorphic to K equals the number of subgraphs of H isomorphic to K. Consequently G and H have the same number of edges. The degree of a vertex i in a graph G is denoted $\deg(i,G)$. By Kelly's lemma, $\deg(i,G)=\deg(i,H)$.

We will often be concerned with the difference between two nearly identical graphs G and G'. The *exclusive or* (also called *symmetric difference*) of graphs G and G' is denoted $G \oplus G' = (G-G') \cup (G'-G)$. If G and G' have the same number of edges, then $|G-G'| = |G'-G|$, so that $|G \oplus G'|$ is even.

1.4 Lemma. *$G^{q_i} \oplus H$ consists of pairs of the form $[i,x]$.*
Proof. This follows because $(G-i)^{p_i} = H-i$.

1.5 Lemma. *If $\deg(i,G) = \deg(i^{q_j},G)$, for some i and j, where $i \neq j$, then $[i,j] \in G$ if and only if $[i^{q_j},j] \in H$.*
Proof. By Kelly's lemma, $\deg(i,G) = \deg(i,H)$ and $\deg(i^{q_j},G) = \deg(i^{q_j},H)$. Since $\deg(i,G) = \deg(i^{q_j},G) = \deg(i^{q_j},H)$ and $\deg(i,G-j) = \deg(i^{q_j},H-j)$, the conclusion follows.

1.6 Corollary. *If $i^{q_j} = i$ for some i and j, where $i \neq j$, then $[i,j] \in G$ if and only if $[i,j] \in H$.*
Proof. If $i^{q_j} = i$, then $\deg(i,G) = \deg(i^{q_j},G)$. The result follows from the previous lemma.

1.7 Corollary. *Given any q_j, $G^{q_j} = H$ if and only if q_j preserves degree.*
Proof. If $G^{q_j} = H$, then $\deg(i,G) = \deg(i^{q_j},H) = \deg(i^{q_j},G)$, so that q_j preserves degree. Conversely, suppose that q_j preserves degree. We have $\deg(i,G) = \deg(i^{q_j},G)$, for all $i \neq j$. It follows that $[i,j] \in G$ if and only if $[i^{q_j},j] \in H$; that is, $G^{q_j} = H$.

The automorphism group of a graph G is denoted $\mathrm{aut}(G)$.

1.8 Lemma. *If $q_i q_j^{-1} \in \mathrm{aut}(G)$ for some $i \neq j$, then $G^{q_i} = G^{q_j} = H$.*
Proof. If $q_i q_j^{-1} \in \mathrm{aut}(G)$, then $G^{q_i q_j^{-1}} = G$, so that $G^{q_i} = G^{q_j}$. We show that $G^{q_i} = H$. Let $[x,y] \in G$. If $i \notin \{x,y\}$, then $[x,y]^{q_i} \in H$. Otherwise, let $[i,x] \in G$. Then since $q_i q_j^{-1} \in \mathrm{aut}(G)$, we have $[i,x]^{q_i q_j^{-1}} = [i^{q_j^{-1}}, x^{q_i q_j^{-1}}] \in G$. If $j \neq x^{q_i q_j^{-1}}$, apply q_j to get $[i^{q_j^{-1}}, x^{q_i q_j^{-1}}]^{q_j} \in H$, or $[i,x]^{q_i} \in H$. Otherwise $x = j^{q_i^{-1}}$. We thus have, that $[x,y] \in G$ if and only if $[x,y]^{q_i} \in H$, for all pairs $[x,y]$, except possibly for the pair $[x,y] = [i, j^{q_i^{-1}}]$. But G and H have the same number of edges. Therefore we also have $[i, j^{q_i^{-1}}] \in G$ if and only if $[i,j] \in H$, so that $G^{q_i} = H$.

1.9 Corollary. *If $q_i = q_j$, for some $i \neq j$, then $G^{q_i} = H$.*
Proof. If $q_i = q_j$, then $q_i q_j^{-1} \in \mathrm{aut}(G)$.

Lemma 1.8 is one of the simplest possible cases of an algebraic relation on the permutations q_i implying the reconstructibility of G; namely $G^{q_iq_j^{-1}} = G$ implies that G is reconstructible. In this paper, we show that reconstructibility can also be implied by a weaker algebraic relation. We consider the case when $q_iq_j^{-1}$ is not an automorphism of G, but where there exists an algebraic relation of the form $G^{q_iq_j^{-1}} = G^{q_kq_\ell^{-1}}$, for suitable vertices i, j, k, and ℓ.

Notice that if $G^{q_iq_j^{-1}} = G^{q_kq_j^{-1}}$, we can apply q_j to each side of the equation and obtain $G^{q_iq_k^{-1}} = G$, from which Lemma 1.8 gives $G \cong H$. Thus, the cases to consider are:

1. $G^{q_iq_j^{-1}} = G^{q_jq_i^{-1}}$, where $i \neq j$;
2. $G^{q_iq_j^{-1}} = G^{q_iq_k^{-1}}$, where i, j, and k are distinct;
3. $G^{q_iq_j^{-1}} = G^{q_kq_i^{-1}}$, where i, j, and k are distinct;
4. $G^{q_iq_j^{-1}} = G^{q_kq_\ell^{-1}}$, where i, j, k, and ℓ are distinct.

Notice that we do not need to consider the case $G^{q_iq_j^{-1}} = G^{q_jq_k^{-1}}$, because relabelling the subscipts i, j, k as k, i, j reduces it to case 3. In this paper we shall consider the first two of these possibilities.

Given any pair $[x, y]$, we can apply $q_iq_j^{-1}$ to it repeatedly to obtain a cyclic sequence of pairs, which we will denote by $\Omega(x, y)$, the *pair-orbit* of $[x, y]$. It has the property that $\Omega(x, y)^{q_iq_j^{-1}} = \Omega(x, y)$. If we conjugate $q_iq_j^{-1}$ by q_i or q_j, we obtain the permutation $q_j^{-1}q_i$. Therefore, if Ω is a pair-orbit of $q_iq_j^{-1}$, then Ω^{q_i} is a pair-orbit of $q_j^{-1}q_i$, and furthermore, $\Omega^{q_i} = \Omega^{q_j}$. If G were composed of a union of pair orbits, $\Omega(x, y)$, then since $\Omega(x, y)^{q_i} = \Omega(x, y)^{q_j}$, we would have $G^{q_i} = G^{q_j}$, so that $G \cong H$, by Lemma 1.8. Therefore there must be at least one pair-orbit $\Omega(x, y)$ that is only partially contained in G.

1.10 Lemma. *Let $\Omega = \Omega(x, y)$ be a pair-orbit of $q_iq_j^{-1}$, and let C be a cycle of $q_iq_j^{-1}$. Then $\deg(u, \Omega) = \deg(v, \Omega)$ for all $u, v \in C$.*
Proof. There are three possibilities: $x, y \in C$; $x \in C$ and $y \notin C$; or $x, y \notin C$. The result follows easily in each case.

2. $G^{q_iq_j^{-1}} = G^{q_jq_i^{-1}}$

We assume throughout this section that $G^{q_iq_j^{-1}} = G^{q_jq_i^{-1}}$, for some $i \neq j$. We have $(q_iq_j^{-1})^2 \in \mathrm{aut}(G)$.

2.1 Lemma. *$q_iq_j^{-1} \in \mathrm{aut}(G \oplus G^{q_iq_j^{-1}})$ and $q_iq_j^{-1} \in \mathrm{aut}(G \cap G^{q_iq_j^{-1}})$ and $q_iq_j^{-1} \in \mathrm{aut}(\overline{G} \cap \overline{G}^{q_iq_j^{-1}})$.*
Proof. Since $G^{q_iq_j^{-1}} = G^{q_jq_i^{-1}}$, we have $(G \oplus G^{q_iq_j^{-1}})^{q_iq_j^{-1}} = G \oplus G^{q_iq_j^{-1}}$. Similarly for $G \cap G^{q_iq_j^{-1}}$ and $\overline{G} \cap \overline{G}^{q_iq_j^{-1}}$.

It follows from Lemma 2.1 that if $[x,y] \in G \oplus G^{q_i q_j^{-1}}$ then $\Omega(x,y) \subseteq G \oplus G^{q_i q_j^{-1}}$; similarly, if $[x,y] \in G \cap G^{q_i q_j^{-1}}$, then $\Omega(x,y) \subseteq G \cap G^{q_i q_j^{-1}}$; and if $[x,y] \in \overline{G} \cap \overline{G}^{q_i q_j^{-1}}$ then $\Omega(x,y) \subseteq \overline{G} \cap \overline{G}^{q_i q_j^{-1}}$.

2.2 Lemma. *Let $[x,y] \in G \oplus G^{q_i q_j^{-1}}$. Then $[x,y]^{p_i p_j^{-1}}$ and $[x,y]^{p_j p_i^{-1}}$ do not exist. Consequently, $\{x,y\} \cap \{i, j^{q_i^{-1}}\} \neq \emptyset$ and $\{x,y\} \cap \{j, i^{q_j^{-1}}\} \neq \emptyset$.*
Proof. If $[x,y] \in G - G^{q_i q_j^{-1}}$, then $[x,y]^{q_i q_j^{-1}} \in (G - G^{q_i q_j^{-1}})^{q_i q_j^{-1}} = G^{q_i q_j^{-1}} - G$. Hence $[x,y]^{p_i p_j^{-1}}$ does not exist, since $[x,y]^{q_i q_j^{-1}} \notin G$. Similarly if $[x,y] \in G^{q_i q_j^{-1}} - G$. By Lemma 1.2, we have $\{x,y\} \cap \{i, j^{q_i^{-1}}\} \neq \emptyset$. Similarly for $[x,y]^{p_j p_i^{-1}}$.

2.3 Lemma. *Let $[x,y] \in G \oplus G^{q_i q_j^{-1}}$, and let $\Omega = \Omega(x,y)$. Then $|\Omega \cap (G - G^{q_i q_j^{-1}})| = |\Omega \cap (G^{q_i q_j^{-1}} - G)|$, and $|\Omega|$ is even.*
Proof. This follows since $(G - G^{q_i q_j^{-1}})^{q_i q_j^{-1}} = G^{q_i q_j^{-1}} - G$.

Case (a). $q_i q_j^{-1}$ is of type 1.

Consider a pair $[x,y] \in G \oplus G^{q_i q_j^{-1}}$. Let C_{ij} denote the cycle of $q_i q_j^{-1}$ containing i and j. Then $i^{q_j^{-1}}$ and $j^{q_i^{-1}}$ are also in C_{ij}. By Lemma 2.2, we have $x, y \in C_{ij}$. If C_{ij} had odd length, then $|\Omega(x,y)|$ would be odd, a contradiction. We conclude that C_{ij} has even length. Now $|C_{ij}| \neq 2$, since this implies that $i^{q_i q_j^{-1}} = j$, so that $i = j$, a contradiction. Therefore C_{ij} has at least 4 vertices, so that it has at least 2 vertices other than $\{i,j\}$. There are two possibilities to consider, either $i^{q_j^{-1}}$ and $j^{q_i^{-1}}$ are distinct vertices, or else $i^{q_j^{-1}} = j^{q_i^{-1}}$.

2.4 Theorem. *If $G^{q_i q_j^{-1}} = G^{q_j q_i^{-1}}$, and $q_i q_j^{-1}$ is of type 1, and $i^{q_j^{-1}} \neq j^{q_i^{-1}}$ then $G \cong H$.*
Proof. We have $[x,y] \in G \oplus G^{q_i q_j^{-1}}$. By Lemma 2.2, every pair of $\Omega(x,y)$ must intersect both $\{i, j^{q_i^{-1}}\}$ and $\{j, i^{q_j^{-1}}\}$. It follows that $|C_{ij}| = 4$. Refer to Figure 2. Write the cycle C_{ij} as $(i, i^{q_j^{-1}}, j^{q_i^{-1}}, j)$. Clearly $\Omega(x,y) = \Omega(i,j)$. Without loss of generality, we can take $\Omega \cap (G - G^{q_i q_j^{-1}}) = \{[i,j], [i^{q_j^{-1}}, j^{q_i^{-1}}]\}$ and $\Omega \cap (G^{q_i q_j^{-1}} - G) = \{[i, i^{q_j^{-1}}], [j, j^{q_i^{-1}}]\}$. Hence, $|G - G^{q_i q_j^{-1}}| = |G^{q_i q_j^{-1}} - G| = 2$. We see that $i, i^{q_j^{-1}}, j^{q_i^{-1}}$ and j all have degree one in $G - G^{q_i q_j^{-1}}$. By Lemma 1.10, they all have the same degree in G, call it α.

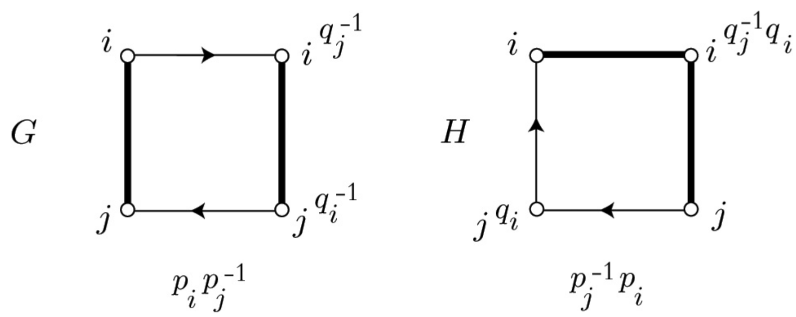

Figure 2, G and H when $|C_{ij}| = 4$ and $i^{q_j^{-1}} \neq j^{q_i^{-1}}$

We now find $(G-i)^{p_i}$ and $(G-j)^{p_j}$ in order to find H. We know that $C_{ij}^{q_i} = (i, i^{q_j^{-1}q_i}, j, j^{q_i})$ is a cycle of $q_j^{-1}q_i$, as shown in Figure 2. If $u \notin C_{ij}$, then $\deg(u,G) = \deg(u^{q_i},H)$, by Lemma 1.10, since $\Omega(i,j)$ is the only pair-orbit that is only partially contained in G. We find that $[i^{q_j^{-1}}, j^{q_i^{-1}}]^{q_i} = [j, i^{q_j^{-1}q_i}] \in H$ and $[i^{q_j^{-1}}, j^{q_i^{-1}}]^{q_j} = [i, i^{q_j^{-1}q_i}] \in H$, since $j^{q_i^{-1}q_j} = i^{q_j^{-1}q_i}$. Therefore $\deg(i^{q_j^{-1}q_i}, H) = \alpha+1$. We also find that $[j, j^{q_i^{-1}}]^{q_i} = [j, j^{q_i}] \notin H$ and $[i, i^{q_j^{-1}}]^{q_j} = [i, j^{q_i}] \notin H$, since $i^{q_j} = j^{q_i}$. It follows that $\deg(j^{q_i},H) = \alpha - 1$. But G and H must have the same degree sequences, a contradiction. We conclude that $G - G^{q_iq_j^{-1}} = G^{q_iq_j^{-1}} - G = \emptyset$, so that $G = G^{q_iq_j^{-1}}$, from which it follows that $G \cong H$, by Lemma 1.8.

The second possibility is when $i^{q_j^{-1}} = j^{q_i^{-1}}$. This is illustrated in Figure 3.

2.5 Theorem. *If $G^{q_iq_j^{-1}} = G^{q_jq_i^{-1}}$, and $q_iq_j^{-1}$ is of type 1, and $i^{q_j^{-1}} = j^{q_i^{-1}}$ then either $G^{q_i} = H$ or else $G^{q_i} \oplus H = \{[i,j],[i,j^{q_i}]\}$. Furthermore, $[i,j] \in G$ if and only if $[i,j] \in H$. If $[i,j] \in G$ then $(G-[i,j])^{q_i} = H - [i,j]$. If $[i,j] \notin G$ then $(G+[i,j])^{q_i} = H + [i,j]$.*

Proof. We have $[x,y] \in G \oplus G^{q_iq_j^{-1}}$, where $x,y \in C_{ij}$. By Lemma 2.2, every pair of $\Omega(x,y)$ must intersect both $\{i, i^{q_j^{-1}}\}$ and $\{j, i^{q_j^{-1}}\}$. It follows that $|C_{ij}| = 4$, that is $C_{ij} = (i, i^{q_j^{-1}}, j, i^{q_jq_i^{-1}})$. Notice that $[j, i^{q_jq_i^{-1}}]^{p_ip_j^{-1}}$ exists, so that by Lemma 2.2, $\Omega([j, i^{q_jq_i^{-1}}]) \not\subseteq G \oplus G^{q_iq_j^{-1}}$. Therefore either $[x,y] = [i,j]$ or else $[x,y] = [i^{q_j^{-1}}, i^{q_jq_i^{-1}}]$, so that we again have $\Omega(x,y) = \Omega(i,j)$. Without loss of generality, we can take $\Omega \cap (G - G^{q_iq_j^{-1}}) = \{[i,j]\}$ and $\Omega \cap (G^{q_iq_j^{-1}} - G) = \{[i^{q_jq_i^{-1}}, i^{q_j^{-1}}]\}$. Hence, $|G - G^{q_iq_j^{-1}}| = |G^{q_iq_j^{-1}} - G| = 1$. By Lemma 1.10, the vertices of C_{ij} all have the same degree in $G - [i,j]$, call it α. Then $\deg(i,G) = \deg(j,G) = \alpha + 1$ and $\deg(i^{q_j^{-1}}, G) = \deg(i^{q_jq_i^{-1}}, G) = \alpha$.

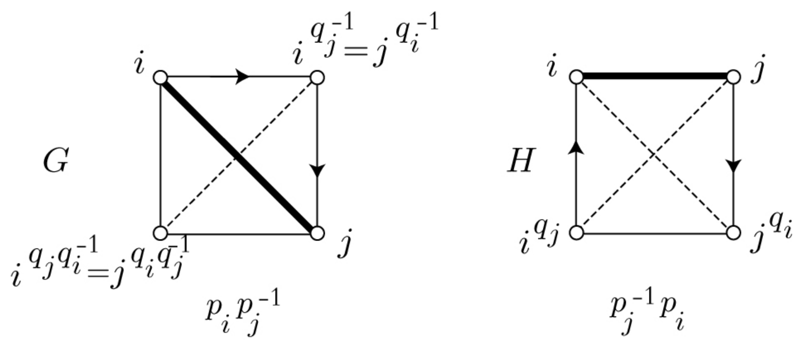

Figure 3, G and H when $|C_{ij}| = 4$ and $i^{q_j^{-1}} = j^{q_i^{-1}}$

We now find $(G-i)^{p_i}$ and $(G-j)^{p_j}$ in order to find H. Since $[i^{q_j^{-1}}, i^{q_jq_i^{-1}}] \notin G$, we find that $[i^{q_j^{-1}}, i^{q_jq_i^{-1}}]^{q_i} = [j, i^{q_j}] \notin H$ and $[i^{q_j^{-1}}, i^{q_jq_i^{-1}}]^{q_j} = [i, j^{q_i}] \notin H$. Therefore $(\Omega(i,j))^{q_i} \cap H = \emptyset$. We know that either $\Omega(i, i^{q_j^{-1}}) \subseteq G$ or $\Omega(i, i^{q_j^{-1}}) \subseteq \overline{G}$. Suppose that $\Omega(i, i^{q_j^{-1}}) \subseteq G$. Then $|(\Omega(i, i^{q_j^{-1}})^{q_i} \cap H| \leq 4$. But $\Omega(i, i^{q_j^{-1}})$ and $\Omega(i,j)$ together contain 5 edges of G. This is impossible, as G and H have the same number of edges. We conclude that $\Omega(i, i^{q_j^{-1}}) \subseteq \overline{G}$. We then find that $[j, j^{q_i^{-1}}]^{q_i} = [j, j^{q_i}] \in \overline{H}$; $[j, i^{q_jq_i^{-1}}]^{q_i} = [i^{q_j}, j^{q_i}] \in \overline{H}$;

and $[i, i^{q_j^{-1}}]^{q_j} = [i, i^{q_j}] \in \overline{H}$. Of the 4 edges of $\Omega(i, i^{q_j^{-1}})^{q_i}$, at most one of them, namely $[i, j]$, can be an edge of H. Since G and H have the same number of edges, we conclude that $[i, j] \in H$. It then follows that $(G - [i, j])^{q_i} = H - [i, j]$. Notice that $\deg(i, H) = \deg(j, H) = \alpha + 1$ and that $\deg(i^{q_j}, H) = \deg(i^{q_j}, H) = \alpha$.

If we do not have $[i, j] \in G - G^{q_i q_j^{-1}}$, then the alternative is $[i^{q_j q_i^{-1}}, i^{q_j^{-1}}] \in G - G^{q_i q_j^{-1}}$. The analysis is very similar. We find that $(\Omega(i, j))^{q_i} \subseteq H$ and that $[j, j^{q_i}], [i^{q_j}, j^{q_i}], [i, i^{q_j}] \in H$ but that $[i, j] \notin H$. It follows that $(G + [i, j])^{q_i} = H + [i, j]$.

Case (b). $q_i q_j^{-1}$ is of type 2.

Let C_i denote the cycle of $q_i q_j^{-1}$ containing i, and C_j the cycle containing j. Refer to the cycle structure of $q_i q_j^{-1}$ illustrated in Figure 1, for a type 2 permutation. Suppose first that $|C_i| \geq 2$ and $|C_j| \geq 2$. By Lemma 2.2, $G \oplus G^{q_i q_j^{-1}} \subseteq \{[i, j], [i, i^{q_j^{-1}}], [j, j^{q_i^{-1}}], [i^{q_j^{-1}}, j^{q_i^{-1}}]\}$. If $[i, i^{q_j^{-1}}] \in G \oplus G^{q_i q_j^{-1}}$, then by Lemma 2.1, $\Omega(i, i^{q_j^{-1}}) \subseteq G \oplus G^{q_i q_j^{-1}}$. Since $i, i^{q_j^{-1}} \in C_i$, we then must have $\Omega(i, i^{q_j^{-1}}) = \{[i, i^{q_j^{-1}}]\}$, which contradicts Lemma 2.3. Similarly if $[j, j^{q_i^{-1}}] \in G \oplus G^{q_i q_j^{-1}}$. It follows that $G \oplus G^{q_i q_j^{-1}} = \{[i, j], [i^{q_j^{-1}}, j^{q_i^{-1}}]\}$. Lemma 2.1 then implies that $j^{q_i q_j^{-1}} = j^{q_i^{-1}}$ so that $C_j = (j, j^{q_i^{-1}})$. Similarly we have $C_i = (i, i^{q_j^{-1}})$, so that $i^{q_j^{-1} q_i} = i^{q_j}$ and $|C_i| = |C_j| = 2$.

2.6 Theorem. *Suppose that $G^{q_i q_j^{-1}} = G^{q_j q_i^{-1}}$, that $q_i q_j^{-1}$ is of type 2, and that $|C_i| = |C_j| = 2$. Then either $G^{q_i} = H$, or else $G^{q_i} \oplus H = \{[i, j], [i, j^{q_i}]\}$.*

Proof. We have $C_i = (i, i^{q_j^{-1}})$ and $C_j = (j, j^{q_i^{-1}})$. If $G \oplus G^{q_i q_j^{-1}} = \emptyset$, then $G^{q_i} = H$, by Lemma 1.8. Otherwise $G \oplus G^{q_i q_j^{-1}} = \{[i, j], [i^{q_j^{-1}}, j^{q_i^{-1}}]\}$. Refer to Figure 4.

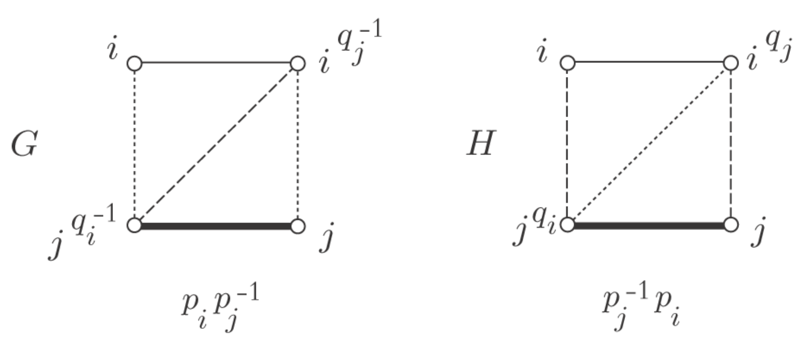

Figure 4, G and H when $|C_i| = |C_j| = 2$

We now calculate $(G - i)^{p_i}$ and $(G - j)^{p_j}$ in order to find H. We have $[i^{q_j^{-1}}, j^{q_i^{-1}}] \in G \oplus G^{q_i q_j^{-1}}$. Therefore $[i^{q_j^{-1}}, j^{q_i^{-1}}]^{p_j} = [i, j^{q_i}] \in G^{q_i} \oplus H$. If $[i, u] \in G^{q_i} \oplus H$, where $u \neq j$, then $[i, u]^{p_j^{-1}} = [i^{q_j^{-1}}, u^{q_j^{-1}}] \in G \oplus G^{q_i q_j^{-1}}$. It follows that $u^{q_j^{-1}} = j^{q_i^{-1}}$, so that $u = j^{q_i}$. Since $G^{q_i} \oplus H$ must be even, we conclude that $G^{q_i} \oplus H = \{[i, j], [i, j^{q_i}]\}$. $\|$

It follows from Theorem 2.6, that either $[i, j] \in G$ and $[i^{q_j^{-1}}, j^{q_i^{-1}}] \notin G$, in which case $[i, j] \in H$ and $[i, j^{q_i}] \notin H$, or else vice versa. Notice the following:

- $[i,j] \in G$ if and only if $[i,j] \in H$
- $[i, i^{q_j^{-1}}] \in G$ if and only if $[i, i^{q_j^{-1}}]^{p_j} = [i, i^{q_j}] \in H$
- $[j, j^{q_i^{-1}}] \in G$ if and only if $[j, j^{q_i^{-1}}]^{p_i} = [j, j^{q_i}] \in H$
- $[i^{q_j^{-1}}, j^{q_i^{-1}}] \in G$ if and only if $[i, j^{q_i}] \in H$ and $[j, i^{q_j}] \in H$
- $[i, j^{q_i^{-1}}] \in G$ and $[j, i^{q_j^{-1}}] \in G$ if and only if $[i^{q_j}, j^{q_i}] \in H$

We can construct graphs G and H satisfying Theorem 2.6, as shown in Figure 5. Since G and H have the same number of edges, there are only two possibilities for the four pairs connecting C_i and C_j. Either only $[i,j] \in G$ and $[i,j] \in H$, or else $[i, j^{q_i^{-1}}], [j, i^{q_j^{-1}}], [i^{q_j^{-1}}, j^{q_i^{-1}}] \in G$, and $[i, j^{q_i}], [j, i^{q_j}], [i^{q_j}, j^{q_i}] \in H$. In Figure 5, the cycles of $q_i q_j^{-1}$ are represented by ellipses of vertices that are alternately white and black. There may be other edges between the cycles, and there may be additional cycles, as well. It is evident from the construction that $(G-i)^{q_i} = H - i$ and that $(G-j)^{q_j} = H - j$, and that $|C_i| = |C_j| = 2$. In this example, G and H are isomorphic. However, this is not necessarily so, as we can add additional edges between the two cycles, as well as additional cycles. The resulting graphs would then not be hypomorphic, although we do not know how to prove this in general.

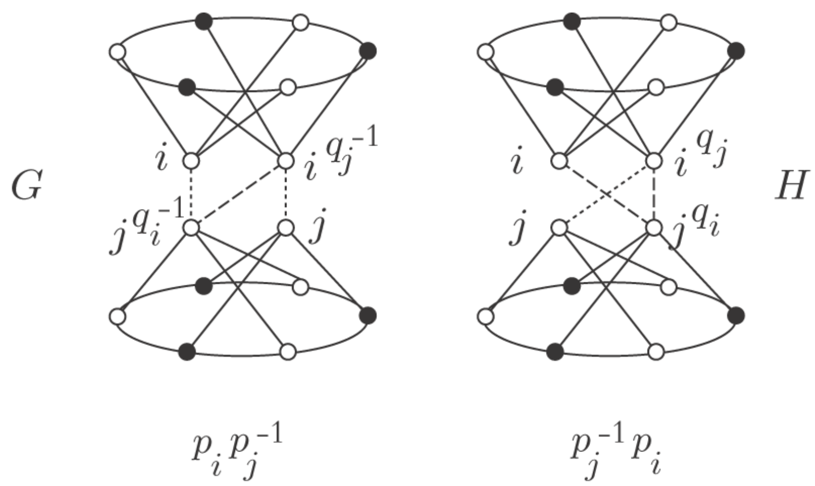

Figure 5, Possible graphs G and H when $|C_i| = |C_j| = 2$

Consider now the case when $|C_i| = 1$ and $|C_j| \geq 2$. Since $i^{q_j} = i$, by Corollary 1.6, we have $[i,j] \in G$ if and only if $[i,j] \in H$. Now $[i,j] \in G$ if and only if $[i,j]^{q_j} = [i,j] \in G^{q_j}$. Therefore $[i,j] \notin G^{q_j} \oplus H$. Suppose that $[j,u] \in G^{q_j} \oplus H$. Since $u \neq i$, we have $[j,u]^{p_i^{-1}} \in G \oplus G^{q_i q_j^{-1}}$. By Lemma 2.2, it follows that $\{j^{q_i^{-1}}, u^{q_i^{-1}}\} \cap \{i, j^{q_i^{-1}}\} \neq \emptyset$ and $\{j^{q_i^{-1}}, u^{q_i^{-1}}\} \cap \{i,j\} \neq \emptyset$. The second condition implies that $u = j^{q_i}$. Hence $G^{q_j} \oplus H \subseteq \{[j, j^{q_i}]\}$. But since $|G^{q_j} \oplus H|$ is even, it follows that $G^{q_j} = H$. We summarise the previous results in the following theorem.

2.7 Theorem. *Suppose that $G^{q_i q_j^{-1}} = G^{q_j q_i^{-1}}$, and that $q_i q_j^{-1}$ is of type 2. If $|C_i| \neq |C_j|$ or if $|C_i| = |C_j| \geq 3$ then $G \cong H$. If $|C_i| = |C_j| = 2$, then either $G^{q_i} = H$ or else $G^{q_i} \oplus H = \{[i,j], [i, j^{q_i}]\}$.*

There remains the situation when $|C_i| = |C_j| = 1$. We then have $i^{q_j} = i$ and $j^{q_i} = j$. Lemma 2.2 now only says that $[x, y] \in G \oplus G^{q_i q_j^{-1}}$ satisfies $\{x, y\} \cap \{i, j\} \neq \emptyset$. Since $q_i q_j^{-1}$ fixes both i and j, it acts as an automorphism of $G - \{i, j\}$, and so $(G - \{i, j\})^{q_i} = (G - \{i, j\})^{q_j} = H - \{i, j\}$. The vertices of G are organized into cycles of $q_i q_j^{-1}$. Since $(q_i q_j^{-1})^2 \in \mathrm{aut}(G)$, we know that if i or j is joined to a vertex on a cycle of $q_i q_j^{-1}$, then it is also joined to every second vertex on the cycle. We can construct graphs G and H satisfying these requirements, as shown in Figure 6. In the diagram on the left, the cycles of $q_i q_j^{-1}$ are represented by ellipses of vertices that are alternately white and black. It is evident from the construction that $(G - i)^{q_i} = H - i$ and that $(G - j)^{q_j} = H - j$, and that $|C_i| = |C_j| = 1$. In this example, G and H are not hypomorphic. However, we do not know how to prove this in general. It may even be that $(q_i q_j^{-1})^2$ is the identity, so that the cycles of $q_i q_j^{-1}$ have length at most two. This case is illustrated in the diagram on the right. It is very easy to see that the graphs of this example are not hypomorphic, but more complicated examples could be constructed.

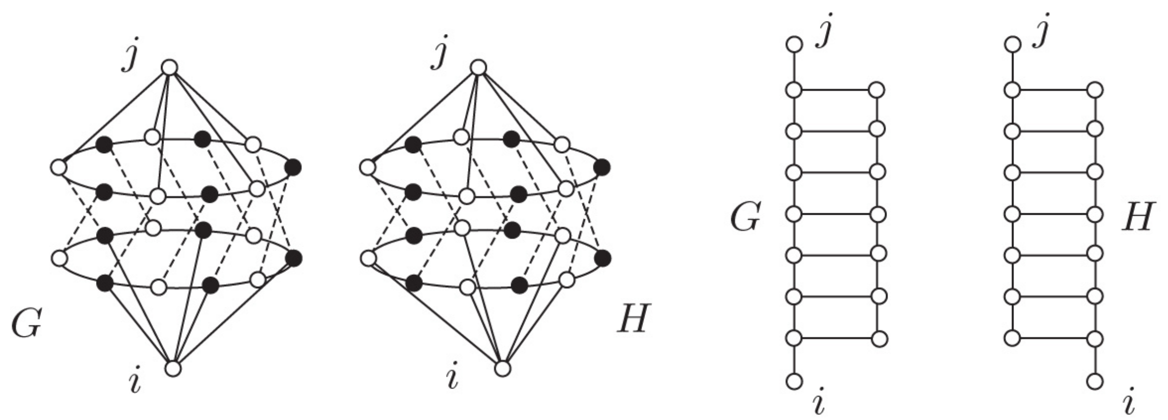

Figure 6, Possible graphs G and H when $|C_i| = |C_j| = 1$

3. $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$, where i, j, and k are distinct.

We assume throughout this section that $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$. Let $G' = G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$. Notice that $(G')^{q_j q_k^{-1}} = G^{q_i q_j^{-1} q_j q_k^{-1}} = G^{q_i q_k^{-1}} = G'$. This gives:

3.1 Lemma. *Let $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$, where i, j, and k are distinct. Then $q_j q_k^{-1} \in \mathrm{aut}(G')$.*

Since $p_i p_j^{-1}$ is a partial automorphism of G, it follows that G and G' are nearly identical. We are concerned with $G \oplus G'$, the difference between G and G'. Suppose that $[x, y] \in G \oplus G'$. If $[x, y]^{p_j p_i^{-1}}$ existed, then we could apply $p_i p_j^{-1}$ to it. Since $p_i p_j^{-1}$ is a partial automorphism of G, and since $q_i q_j^{-1}$ maps edges of G to edges of G', we obtain $[x, y]_G = [x, y]_{G'}$, a contradiction. It follows that $[x, y]^{p_j p_i^{-1}}$ does not exist, so that by Lemma 1.2, $\{x, y\} \cap \{j, i^{q_j^{-1}}\} \neq \emptyset$. Since G' can also be written as $G^{q_i q_k^{-1}}$, we conclude that $\{x, y\} \cap \{k, i^{q_k^{-1}}\} \neq \emptyset$ is also a requirement. This gives:

3.2 Lemma. *If* $\{j, i^{q_j^{-1}}\}$ *and* $\{k, i^{q_k^{-1}}\}$ *are disjoint, then* G *and* G' *can differ in at most 4 pairs,* $[j,k], [j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}]$*, and* $[i^{q_j^{-1}}, i^{q_k^{-1}}]$.

We will prove that when $\{j, i^{q_j^{-1}}\}$ and $\{k, i^{q_k^{-1}}\}$ are disjoint, G and H are always isomorphic; that is, the algebraic condition $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$ implies the reconstructibility of G, when $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 4$. The proof consists of a number of cases, determined by the cycle structure of the permutation $q_j q_k^{-1}$. We will prove that in each case, H is equal to one of G^{q_i}, G^{q_j}, or G^{q_k}.

Notice that the graphs $G, G', H, G^{q_i}, G^{q_j}$, and G^{q_k} all have the same number of edges. Therefore any symmetric difference of the form $G \oplus G'$, $G \oplus H$, $G^{q_i} \oplus H$, etc., all have an even number of pairs, and that $|G - G'| = |G' - G|$, etc. We can conclude that if $G \oplus G'$ has four pairs, that $|G - G'| = |G' - G| = 2$.

The technique used in each case is based on the following idea: Given $G \oplus G'$, we apply the mappings p_j and p_k to its pairs. Applying p_j and p_k to pairs of G will result in pairs of H. Since $G' = G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$, applying p_j or p_k to pairs of G' will result in pairs of G^{q_i}. Therefore, applying p_j and p_k to pairs of $G - G'$ will give pairs of $H - G^{q_i}$ and applying p_j and p_k to pairs of $G' - G$ will give pairs of $G^{q_i} - H$. So given $G \oplus G'$, we can often obtain $G^{q_i} \oplus H$.

We look at the cycle structure of $q_j q_k^{-1}$. We assume that G and H are non-isomorphic, hypomorphic graphs, and obtain a contradiction in each case.

3.3 Lemma. $G^{q_i} \oplus H$ *consists of all pairs* $[x,y]^{p_j}$ *and* $[x,y]^{p_k}$ *which exist, such that* $[x,y] \in G \oplus G'$.

Proof. Recall that q_j and q_k both map G' to G^{q_i}, and that p_j and p_k both map pairs of G to H. Therefore, given $[x,y] \in G \oplus G'$, if $[x,y]^{p_j}$ exists, then it is a pair of $G^{q_i} \oplus H$, and similarly for $[x,y]^{p_k}$. Conversely, if $[i,w] \in G^{q_i} \oplus H$, then at least one of $[i,w]^{p_j^{-1}}$ and $[i,w]^{p_k^{-1}}$ exists, since $j \neq k$, giving a pair $[x,y] \in G \oplus G'$ with the required properties.

3.4 Lemma. *If* (j) *or* (k) *is a cycle of* $q_j q_k^{-1}$*, then* $|G \oplus G'| \neq 4$.

Proof. Suppose that (j) is a cycle of $q_j q_k^{-1}$. We have $j^{q_j q_k^{-1}} = j$, so that $j^{q_k} = j$. By Lemma 1.6, $[j,k] \in G$ if and only if $[j,k] \in H$. We also know that $[j,k] \in G'$ if and only if $[j,k]^{q_k} = [j,k] \in (G')^{q_k} = G^{q_i}$. It then follows that $[j,k] \in G \oplus G'$ if and only if $[j,k] \in G^{q_i} \oplus H$, a contradiction to Lemma 1.4. Therefore $[j,k] \notin G \oplus G'$, so that $|G \oplus G'| \neq 4$, by Lemma 3.2. The proof is similar if (k) is a cycle.

3.5 Theorem. $|G \oplus G'| = 4$ *is impossible.*

Proof. Suppose that $|G \oplus G'| = 4$. By the preceding lemmas, we have $G \oplus G' = \{[j,k], [j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}], [i^{q_j^{-1}}, i^{q_k^{-1}}]\}$. We also know that (j) and (k) are not cycles of $q_j q_k^{-1}$. Therefore $k^{p_j p_k^{-1}}$ exists, whether $q_j q_k^{-1}$ is of type 1 or 2. Consider the pair $[k, i^{q_j^{-1}}] \in G \oplus G'$. Without loss of generality, we can assume that $[k, i^{q_j^{-1}}] \in G' - G$ (by considering $\overline{G}$ instead of G, if necessary). Apply $p_j p_k^{-1}$ to get $[k^{q_j q_k^{-1}}, i^{q_k^{-1}}] \in G' - G$, since $q_j q_k^{-1}$ is an automorphism of G', and $p_j p_k^{-1}$ is a partial automorphism of G. Comparing $[k^{q_j q_k^{-1}}, i^{q_k^{-1}}]$ with the four pairs of $G \oplus G'$, we conclude that either $k^{q_j q_k^{-1}} = j$ or $i^{q_j^{-1}}$.

Suppose first that $k^{q_j q_k^{-1}} = i^{q_j^{-1}}$. We have two pairs, $[k, i^{q_j^{-1}}], [i^{q_j^{-1}}, i^{q_k^{-1}}] \in G' - G$. The remaining two pairs of $G \oplus G'$ are $[j,k], [j, i^{q_k^{-1}}] \in G - G'$. We now apply $p_k p_j^{-1}$ to $[j, i^{q_k^{-1}}]$ to obtain $[j^{p_k p_j^{-1}}, i^{q_j^{-1}}]$. Now this pair must exist, since (j) and (k) are not cycles of $q_j q_k^{-1}$. Therefore it is a third pair of $G - G'$, which is impossible.

Consequently, we must have $k^{q_j q_k^{-1}} = j$. It follows that $q_j q_k^{-1}$ is of type 1. We have two pairs, $[k, i^{q_j^{-1}}], [j, i^{q_k^{-1}}] \in G' - G$. Therefore $[j,k], [i^{q_j^{-1}}, i^{q_k^{-1}}] \in G - G'$. Apply $p_j p_k^{-1}$ and $p_k p_j^{-1}$ to $[i^{q_j^{-1}}, i^{q_k^{-1}}]$, to get $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}]$ and $[i^{q_j^{-1}}, i^{q_j^{-1} p_k p_j^{-1}}]$. We must consider whether these pairs exist or not.

Suppose first that $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}]$ and $[i^{q_j^{-1}}, i^{q_j^{-1} p_k p_j^{-1}}]$ do not exist. Then by Lemma1.2, $i^{q_k^{-1}} = j$ or $k^{p_j^{-1}}$ and $i^{q_j^{-1}} = k$ or $j^{p_k^{-1}}$. As $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 4$, we must have $i^{q_k^{-1}} = k^{q_j^{-1}}$ and $i^{q_j^{-1}} = j^{q_k^{-1}}$. This is illustrated in Figure 7. We find that the cycle of $q_j q_k^{-1}$ containing j and k has exactly 4 vertices.

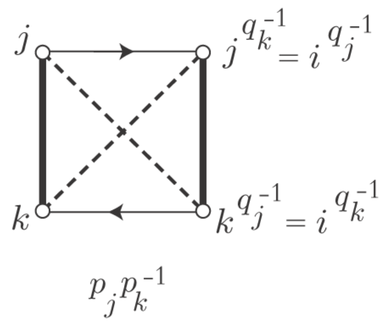

Figure 7, $|G \oplus G'| = 4$, $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}]$ does not exist

We now calculate $G^{q_i} \oplus H$, using Lemma 3.3, by applying p_j and p_k to pairs of $G \oplus G'$. The result is $G^{q_i} \oplus H = \{[i, j^{q_k}], [i,j], [i,k]\}$, since $j^{q_k} = k^{q_j}$. This is impossible, as $|G^{q_i} \oplus H|$ must be even.

There remains the situation when $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}]$ or $[i^{q_j^{-1}}, i^{q_j^{-1} p_k p_j^{-1}}]$ exists. Suppose that $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}]$ exists. As this pair must be in $G - G'$, we conclude that $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}] = [i^{q_j^{-1}}, i^{q_k^{-1}}]$. It follows that $(i^{q_k^{-1}})^{q_j q_k^{-1}} = i^{q_j^{-1}}$, or equivalently, $i^{(q_k^{-1} q_j)^2} = i$. This situation is illustrated in Figure 8.

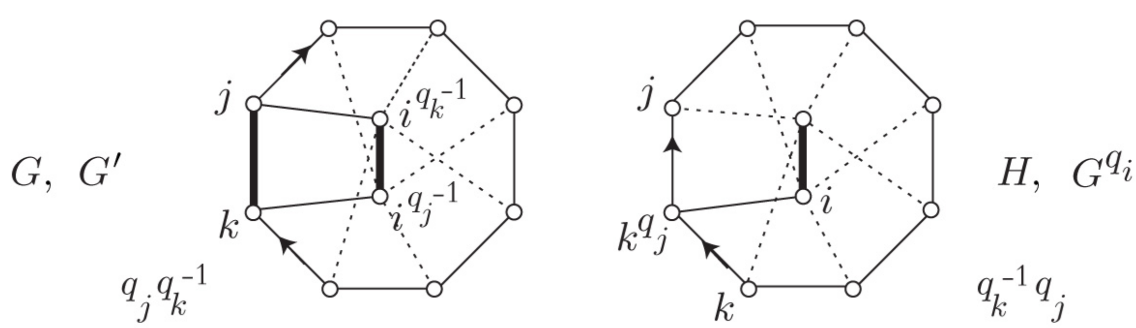

Figure 8, $|G \oplus G'| = 4$, $[i^{q_k^{-1}}, i^{q_k^{-1} p_j p_k^{-1}}] = [i^{q_j^{-1}}, i^{q_k^{-1}}]$

Let C denote the cycle of $q_j q_k^{-1}$ containing j and k. Since $q_j q_k^{-1} \in \mathrm{aut(G')}$, the vertices of C all have the same degree in G', call it α. The cycle of $q_j q_k^{-1}$ containing

$i^{q_j^{-1}}$ is $(i^{q_j^{-1}}, i^{q_k^{-1}})$. Let β denote the degree of $i^{q_j^{-1}}$ and $i^{q_k^{-1}}$ in G'. Now $G' - G = \{[k, i^{q_j^{-1}}], [j, i^{q_k^{-1}}]\}$ and $G - G' = \{[j,k], [i^{q_j^{-1}}, i^{q_k^{-1}}]\}$, ie, G is formed from G' by removing the edges $[k, i^{q_j^{-1}}], [j, i^{q_k^{-1}}]$ and adding the edges $[j,k], [i^{q_j^{-1}}, i^{q_k^{-1}}]$. These pairs are indicated in the diagram. Consequently, all vertices of C also have degree α in G; and $i^{q_j^{-1}}$, $i^{q_k^{-1}}$ both have degree β in G. It follows that $\deg(v,G) = \deg(v,G')$, for all $v \in V$. But $G' = G^{q_i q_j^{-1}}$, so that $\deg(v,G) = \deg(v^{q_i q_j^{-1}}, G^{q_i q_j^{-1}}) = \deg(v^{q_i q_j^{-1}}, G') = \deg(v^{q_i q_j^{-1}}, G)$, ie, v and $v^{q_i q_j^{-1}}$ have the same degree in G, for all $v \in V$.

We show that $\alpha = \beta$. Consider the pair-orbit $\Omega(i^{q_j^{-1}}, i^{q_k^{-1}})$ with respect to $q_i q_j^{-1}$. It consists of a cyclic sequence of pairs $(P_0, P_1, \ldots, P_m)$, where $P_0 = [i^{q_j^{-1}}, i^{q_k^{-1}}]$, and $P_{\ell+1} = P_\ell^{q_i q_j^{-1}}$, where $\ell = 0, 1, 2, \ldots, m$, and addition of subscripts is reduced modulo $m+1$. As $G' = G^{q_i q_j^{-1}}$, we conclude that $P_\ell \in G$ if and only if $P_{\ell+1} \in G'$. Now $P_0 \in G - G'$. Therefore $P_m \notin G$. Hence, there is some $\ell \in \{0, 1, \ldots, m-1\}$, such that $P_\ell \in G$ but $P_{\ell+1} \notin G$. Then $P_{\ell+1} \in G' - G$. It follows that $P_{\ell+1}$ is either $[k, i^{q_j^{-1}}]$ or $[j, i^{q_k^{-1}}]$. Now $\deg(i^{q_j^{-1}}, G) = \deg(i^{q_k^{-1}}, G) = \beta$. As $\deg(v,G) = \deg(v^{q_i q_j^{-1}}, G)$, we conclude that $\alpha = \deg(j,G) = \deg(k,G) = \deg(i^{q_j^{-1}}, G) = \deg(i^{q_k^{-1}}, G) = \beta$.

We now find $H - G^{q_i} = \{[i, i^{q_k^{-1} q_j}]\}$ and $G^{q_i} - H = \{[i, k^{q_j}]\}$, using Lemma 3.3. Therefore, if $v \notin \{i^{q_k^{-1} q_j}, k^{q_j}\}$, then $\deg(v, G^{q_i}) = \deg(v, H)$. In G^{q_i}, the vertices of C^{q_i} all have degree α; furthermore i and $i^{q_k^{-1} q_j}$ have degree β, which equals α. Now H is formed from G^{q_i} by removing the edge $[i, k^{q_j}]$ and adding the edge $[i, i^{q_k^{-1} q_j}]$. Hence, in H, the vertices of C^{q_i} all have degree α, except for k^{q_j}, which has degree $\alpha - 1$. Also, $\deg(i, H) = \alpha$, but $\deg(i^{q_k^{-1} q_j}, H) = \alpha + 1$. But G and H must have the same degree sequences, a contradiction.

This completes the proof that $|G \oplus G'| = 4$ is impossible.

We now turn to the situation when $|G \oplus G'| = 2$. There are 6 ways of choosing two edges out of $\{[j,k], [j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}], [i^{q_j^{-1}}, i^{q_k^{-1}}]\}$. However, due to the symmetry between j and k, some of these will be equivalent. We look at each case in turn.

3.6 Lemma. *If $|G \oplus G'| = 2$, then $[j,k] \notin G \oplus G'$.*
Proof. Suppose that $[j,k] \in G - G'$ and that some pair $[x,y] \in G' - G$. Then since $[j,k]^{p_j}$ and $[j,k]^{p_k}$ do not exist, we find that $H - G^{q_i} = \emptyset$ and that $G^{q_i} - H \neq \emptyset$, which is impossible. A similar result holds if $[j,k] \in G' - G$ and $[x,y] \in G - G'$.

3.7 Lemma. *If $G \oplus G' = \{[j, i^{q_k^{-1}}], [i^{q_j^{-1}}, i^{q_k^{-1}}]\}$, then $G^{q_k} = H$.*
Proof. Suppose that $[j, i^{q_k^{-1}}] \in G - G'$ and that $[i^{q_j^{-1}}, i^{q_k^{-1}}] \in G' - G$. Apply p_j and p_k to obtain $[i, j^{q_k}] \in H - G^{q_i}$ and $[i, i^{q_k^{-1} q_j}], [i, i^{q_j^{-1} q_k}] \in G^{q_i} - H$. Since $|H - G^{q_i}| = |G^{q_i} - H| = 1$, we conclude that $i^{q_k^{-1} q_j} = i^{q_j^{-1} q_k}$. It then follows that $H \oplus G^{q_i} = (G \oplus G')^{q_k}$. But this equals $G^{q_k} \oplus G^{q_i}$. Cancelling G^{q_i} leaves $G^{q_k} = H$.

Interchanging j and k in the above lemma gives:

3.8 Lemma. *If* $G \oplus G' = \{[k, i^{q_j^{-1}}], [i^{q_j^{-1}}, i^{q_k^{-1}}]\}$, *then* $G^{q_j} = H$.

There remains one case.

3.9 Theorem. $G \oplus G' = \{[j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}]\}$ *is impossible.*

Proof. Without loss of generality, suppose that $[j, i^{q_k^{-1}}] \in G - G'$ and that $[k, i^{q_j^{-1}}] \in G' - G$. If $[k, i^{q_j^{-1}}]^{p_j p_k^{-1}}$ were to exist, it would also be a pair of $G' - G$, as $p_j p_k^{-1}$ is a partial automorphism of G and an automorphism of G'; that is, we would have $[i^{q_k^{-1}}, k^{q_j q_k^{-1}}] \in G' - G$. This would make $|G' - G| > 1$, which is impossible. We conclude that $[k, i^{q_j^{-1}}]^{p_j p_k^{-1}}$ does not exist. This implies that $k^{p_j p_k^{-1}}$ does not exist; hence $k^{p_j} = k$. Similarly, we find $j^{p_k} = j$. This is illustrated in Figure 9, which shows the cycles (j) and (k) of $q_j q_k^{-1}$, and the cycle C containing $i^{q_j^{-1}}$ and $i^{q_k^{-1}}$. It follows from Lemma 3.3 that $H - G^{q_i} = \{[i, j]\}$ and $G^{q_i} - H = \{[i, k]\}$.

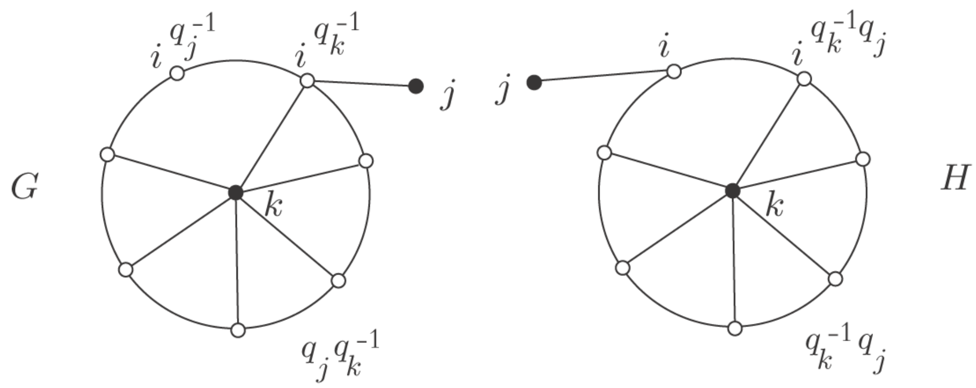

Figure 9, $G \oplus G' = \{[j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}]\}$

We show that G and H have different degree sequences, by observing that G and G^{q_j} have the same degree sequences, but that G^{q_j} and H have different degree sequences. We have $G \oplus G' = \{[j, i^{q_k^{-1}}], [k, i^{q_j^{-1}}]\}$, and $G^{q_i} \oplus H = \{[i, j], [i, k]\}$. If $v \notin \{j, k, i^{q_k^{-1}}, i^{q_j^{-1}}\}$, then $\deg(v, G) = \deg(v, G')$ so that $\deg(v^{q_j}, G^{q_j}) = \deg(v^{q_j}, (G')^{q_j}) = \deg(v^{q_j}, G^{q_i})$. Since $v^{q_j} \notin \{i, j, k\}$, we have $\deg(v, G) = \deg(v^{q_j}, G^{q_i}) = \deg(v^{q_j}, H)$. Since $j^{q_j} = j$ and $k^{q_j} = k$, we also have $\deg(v, G) = \deg(v^{q_j}, H)$ when $v \in \{j, k\}$. We must still compare $\deg(v, G)$ and $\deg(v^{q_j}, H)$ when $v \in \{i^{q_j^{-1}}, i^{q_k^{-1}}\}$.

Now $G' - G = \{[k, i^{q_j^{-1}}]\}$. Since $q_j q_k^{-1} \in \mathrm{aut(G')}$, it follows that in G', k is adjacent to every vertex of C. In G', all vertices of C have a common degree, call it α. Then $\deg(i^{q_j^{-1}}, G') = \deg(i^{q_k^{-1}}, G') = \alpha$. Since $G - G' = \{[j, i^{q_k^{-1}}]\}$, we conclude that $\deg(i^{q_j^{-1}}, G) = \deg(i^{q_j^{-1}}, G') - 1 = \alpha - 1$, and that $\deg(i^{q_k^{-1}}, G) = \deg(i^{q_k^{-1}}, G') + 1 = \alpha + 1$. Then $\deg(i^{q_j^{-1}}, G) = \alpha - 1 = \deg(i, G^{q_i}) - 1$. Since $G^{q_i} \oplus H = \{[i, j], [i, k]\}$, we have $\deg(i, H) = \deg(i, G^{q_i}) = \alpha$. Similarly, $\deg(i^{q_k^{-1}}, G) = \alpha = \deg(i^{q_k^{-1} q_j}, G^{q_i}) = \deg(i^{q_k^{-1} q_j}, H)$. Hence $\deg(i^{q_k^{-1}}, G) = \deg(i^{q_k^{-1} q_j}, H)$. We conclude that G and H have different degree sequences, a contradiction.

We summarise the preceding lemmas and theorems as the following:

3.10 Theorem. *Let G and H be hypomorphic graphs. If $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$, where i, j, and k are distinct, and $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 4$, then H is equal to one of G^{q_i}, G^{q_j}, or G^{q_k}.*

We thus have an algebraic condition forcing the reconstructibility of G. We must still deal with the case when $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| \leq 3$.

4. $|\{j,\ k,\ i^{q_j^{-1}},\ i^{q_k^{-1}}\}| \leq 3$

The possible cases are:

1. $j = i^{q_k^{-1}}$ (or symmetrically, $k = i^{q_j^{-1}}$)
2. $i^{q_j^{-1}} = i^{q_k^{-1}}$
3. $j = i^{q_k^{-1}}$ and $k = i^{q_j^{-1}}$

Case 1. $j = i^{q_k^{-1}}$

In this section we assume that $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 3$ and that $j = i^{q_k^{-1}}$.

4.1 Lemma. *$G \oplus G'$ consists of pairs of the form $[j, x]$ or $[k, i^{q_j^{-1}}]$.*
Proof. As in Lemma 3.2, we find that if $[x, y] \in G \oplus G'$, then $[x, y]^{p_j p_i^{-1}}$ and $[x, y]^{p_k p_i^{-1}}$ do not exist. Hence $\{x, y\} \cap \{j, i^{q_j^{-1}}\} \neq \emptyset$ and $\{x, y\} \cap \{k, i^{q_k^{-1}}\} \neq \emptyset$. Since $j = i^{q_k^{-1}}$, this reduces to either $j \in \{x, y\}$, or $[x, y] = [k, i^{q_j^{-1}}]$.

Notice that $q_j q_k^{-1}$ maps $i^{q_j^{-1}}$ to $i^{q_k^{-1}} = j$ so that $q_j q_k^{-1}$ does not fix the point j.

4.2 Lemma. *Given $[j, x] \in G \oplus G'$, for some x. If $[j, x]^{p_k p_j^{-1}}$ exists, then $k^{q_j} \neq k$ and $G \oplus G' = \{[j, k], [j, j^{q_k^{-1}}], [k, i^{q_j^{-1}}], [j, k^{q_j q_k^{-1}}]\}$.*
Proof. We have $j^{q_k q_j^{-1}} = i^{q_j^{-1}}$. Since $p_k p_j^{-1}$ is both an automorphism of G' and a partial automorphism of G, we have $[j, x]^{p_k p_j^{-1}} = [i^{q_j^{-1}}, x^{q_k q_j^{-1}}] \in G \oplus G'$. By Lemma 4.1, we find that either $x^{q_k q_j^{-1}} = k$ or j. Now we can not have $x^{q_k q_j^{-1}} = j$, because then $x = j^{q_k^{-1}}$, but then $x^{p_k p_j^{-1}}$ does not exist, contrary to assumption. It follows that $x = k^{q_j q_k^{-1}}$ and that $k^{q_j} \neq k$, since $x^{p_k p_j^{-1}}$ exists.

We have found that $[j, k^{q_j q_k^{-1}}], [k, i^{q_j^{-1}}] \in G \oplus G'$, and that these pairs are either both in $G - G'$, or both in $G' - G$. There must be at least two more pairs to balance these, and they must be pairs $[j, x]$, such that $x^{p_k p_j^{-1}}$ does not exist. The only possibilities are $[j, k]$ and $[j, j^{q_k^{-1}}]$. This completes the proof.

Notice that it follows from Lemma 4.2 that $G - G'$ and $G' - G$ are $\{[j, k^{q_j q_k^{-1}}], [k, i^{q_j^{-1}}]\}$ and $\{[j, k], [j, j^{q_k^{-1}}]\}$. Let C_j denote the cycle of $q_j q_k^{-1}$ containing j, and C_k the cycle containing k.

4.3 Theorem. *If* $G \oplus G' = \{[j,k],[j,j^{q_k^{-1}}],[k,i^{q_j^{-1}}],[j,k^{q_j q_k^{-1}}]\}$ *and* $k^{q_j} \neq k$, *then* $|C_j| = 2$, $q_j q_k^{-1}$ *must be of type 2, and* $G^{q_i} \oplus H = \{[i,j],[i,k^{q_j}]\}$.

Proof. Without loss of generality, let $G - G' = \{[k,i^{q_j^{-1}}],[j,k^{q_j q_k^{-1}}]\}$ and $G' - G = \{[j,k],[j,j^{q_k^{-1}}]\}$. Apply p_j and p_k to obtain $H - G^{q_i} = \{[i,k^{q_j}]\}$ and $G^{q_i} - H = \{[i,j]\}$. The case when $q_j q_k^{-1}$ is of type 1 is illustrated in Figure 10; type 2 is illustrated in Figure 11. A dotted line indicates a pair that is known not to be an edge of G.

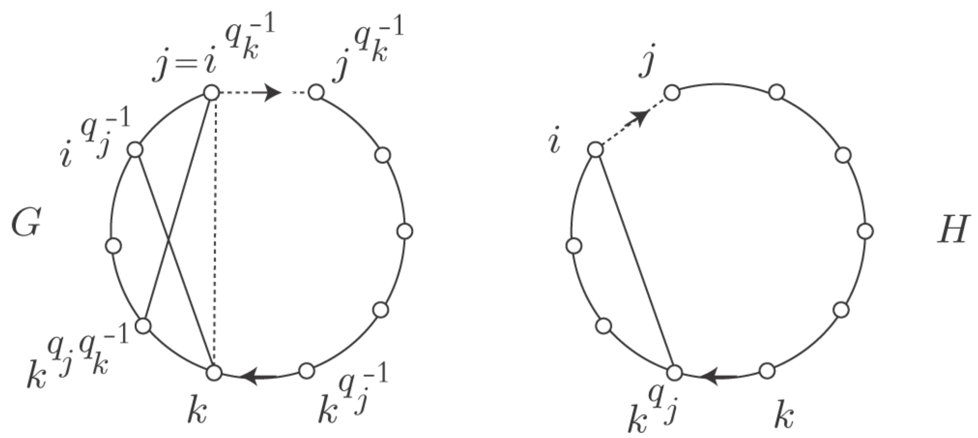

Figure 10, $G \oplus G' = \{[j,k],[j,j^{q_k^{-1}}],[k,i^{q_j^{-1}}],[j,k^{q_j q_k^{-1}}]\}$, $q_j q_k^{-1}$ of type 1

Notice that $k \neq i^{q_j^{-1}}$, by assumption. We also know that $i^{q_j^{-1}} \neq k^{q_j q_k^{-1}}$, for if these two were equal, then $q_j q_k^{-1}$ would map $[j,k^{q_j q_k^{-1}}] \in G - G'$ to $[j,j^{q_k^{-1}}] \in G' - G$, which is impossible, as $q_j q_k^{-1} \in \mathrm{aut(G')}$. We have $j, i^{q_j^{-1}}, j^{q_k^{-1}} \in C_j$; therefore these vertices all have the same degree in G', call it α. Similarly, $k, k^{q_j q_k^{-1}} \in C_k$, so that these vertices also have the same degree in G', call it β. If $q_j q_k^{-1}$ is of type 1, then $C_j = C_k$ and $\alpha = \beta$.

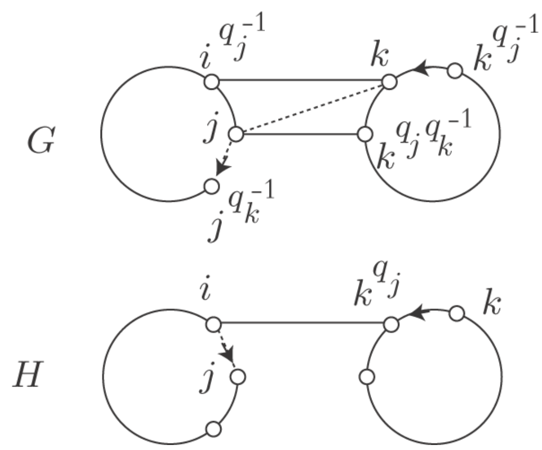

Figure 11, $G \oplus G' = \{[j,k],[j,j^{q_k^{-1}}],[k,i^{q_j^{-1}}],[j,k^{q_j q_k^{-1}}]\}$, $q_j q_k^{-1}$ of type 2

We show that if $|C_j| \geq 3$, then G and H have different degree sequences. This is done by observing that G and G^{q_k} have the same degree sequence, and showing that G^{q_k} and H have different degree sequences. Suppose that $|C_j| \geq 3$, so that $j, j^{q_k^{-1}}$, and $i^{q_j^{-1}}$ are distinct vertices. Since $G - G' = \{[k,i^{q_j^{-1}}],[j,k^{q_j q_k^{-1}}]\}$ and $G' - G = \{[j,k],[j,j^{q_k^{-1}}]\}$, we have $\deg(j, G) = \deg(j^{q_k^{-1}}, G) = \alpha - 1$ and $\deg(i^{q_j^{-1}}, G) = \alpha + 1$. Any other vertices of C_j have degree α in G. Similarly, $\deg(k^{q_j q_k^{-1}}, G) = \beta + 1$, whereas all other vertices of C_k have degree β in G.

If $v \notin \{j, i^{q_j^{-1}}, j^{q_k^{-1}}, k^{q_j q_k^{-1}}\}$, then $\deg(v, G) = \deg(v, G')$ Hence $\deg(v^{q_k}, G^{q_k}) = \deg(v^{q_k}, (G')^{q_k}) = \deg(v^{q_k}, G^{q_i})$. Since $G^{q_i} \oplus H = \{[i, j], [i, k^{q_j}]\}$, we know that $v^{q_k} \notin \{i, j, k^{q_j}\}$, so that $\deg(v, G) = \deg(v^{q_k}, G^{q_i}) = \deg(v^{q_k}, H)$. The vertices of $\{j, i^{q_j^{-1}}, j^{q_k^{-1}}, k^{q_j q_k^{-1}}\}$ have degrees $\{\alpha - 1, \alpha - 1, \alpha, \beta + 1\}$ in G. Since $G^{q_i} - H = \{[i, j]\}$ and $H - G^{q_i} = \{[i, k^{q_j}]\}$, we have $\deg(j^{q_k}, H) = \deg(i, H) = \alpha$; $\deg(i^{q_j^{-1} q_k}, H) = \alpha$; $\deg(j^{q_k^{-1} q_k}, H) = \deg(j, H) = \alpha - 1$; and $\deg(k^{q_j q_k^{-1} q_k}, H) = \deg(k^{q_j}, H) = \beta + 1$. Thus, the degrees of v^{q_k}, when $v \in \{j, i^{q_j^{-1}}, j^{q_k^{-1}}, k^{q_j q_k^{-1}}\}$, are $\{\alpha, \alpha, \alpha - 1, \beta + 1\}$. If $|C_j| \geq 3$, this is not possible. We conclude that $|C_j| = 2$, so that $q_j q_k^{-1}$ is of type 2.

We have been unable to prove that $|C_j| = 2$ is not possible in Theorem 4.3. We find that $i^{q_j^{-1}} = j^{q_k^{-1}}$, so that the degrees in G of $v \in \{j, i^{q_j^{-1}}, k^{q_j q_k^{-1}}\}$ are the same as the degrees in H of v^{q_k}.

Lemmas 4.2 and Theorem 4.3 handle the situation when there is a pair $[j, x] \in G \oplus G'$ such that $[j, x]^{p_k p_j^{-1}}$ exists. We now consider the cases when $[j, x]^{p_k p_j^{-1}}$ does not exist, for every $[j, x] \in G \oplus G'$. Since $j^{p_k p_j^{-1}} = i^{q_j^{-1}}$, we conclude that $x^{p_k p_j^{-1}}$ does not exist. It follows that either $x = k$ or $x = j^{q_k^{-1}}$. The possible pairs for $G \oplus G'$ are $[j, k], [j, j^{q_k^{-1}}]$, and $[k, i^{q_j^{-1}}]$. $G \oplus G'$ contains exactly two of these pairs.

4.4 Lemma. *If $[j, x]^{p_k p_j^{-1}}$ does not exist, for any $[j, x] \in G \oplus G'$, and $[k, i^{q_j^{-1}}] \in G \oplus G'$, then $k^{q_j} = k$.*
Proof. If $[k, i^{q_j^{-1}}]^{p_j p_k^{-1}}$ were to exist, it would equal a pair of $G \oplus G'$, since $p_j p_k^{-1}$ is a partial automorphism of both G and G'. It would follow that $[k, i^{q_j^{-1}}]^{p_j p_k^{-1}} = [j, k^{q_j q_k^{-1}}]$ equals either $[j, k]$ or $[j, j^{q_k^{-1}}]$. This implies that $k^{q_j} = k$.

4.5 Lemma. *$G \oplus G' = \{[j, k], [j, j^{q_k^{-1}}]\}$ is impossible.*
Proof. Calculating $G^{q_i} \oplus H = \{[j, j^{q_k}]\}$ gives a contradiction, since $|G^{q_i} \oplus H|$ must be even.

4.6 Lemma. *$G \oplus G' = \{[j, k], [k, i^{q_j^{-1}}]\}$ is impossible.*
Proof. Calculating $G^{q_i} \oplus H = \{[i, k]\}$, using Lemma 3.3, gives a contradiction, since $|G^{q_i} \oplus H|$ must be even.

4.7 Lemma. *If $G \oplus G' = \{[j, j^{q_k^{-1}}], [k, i^{q_j^{-1}}]\}$, then $G^{q_j} = H$.*
Proof. Without loss of generality, assume that $G - G' = \{[k, i^{q_j^{-1}}]\}$ and that $G' - G = \{[j, j^{q_k^{-1}}]\}$. By Lemma 4.4, $k^{q_j} = k$. Calculate $H - G^{q_i} = \{[i, k]\}$, using Lemma 3.3, and $G^{q_i} - H = \{[j, j^{q_k}]\} = \{[i, j]\}$, since $j^{q_k} = i$. Let C_j be the cycle of $q_j q_k^{-1}$ containing j, as illustrated in Figure 12. In G', all vertices of C_j have the same degree, call it α. Let $\beta = \deg(k, G')$. Suppose first that $|C_j| \geq 3$, so that $i^{q_j^{-1}} \neq j^{q_k^{-1}}$. Since $G - G' = \{[k, i^{q_j^{-1}}]\}$

and $G' - G = \{[j, j^{q_k^{-1}}]\}$, we have $\deg(j, G) = \deg(j^{q_k^{-1}}, G) = \alpha - 1$; $\deg(i^{q_j^{-1}}, G) = \alpha + 1$; and $\deg(k, G) = \beta + 1$. If $v \notin \{j, k, j^{q_k^{-1}}, i^{q_j^{-1}}\}$, then $\deg(v, G) = \deg(v, G')$ so that $\deg(v^{q_k}, G^{q_k}) = \deg(v^{q_k}, (G')^{q_k}) = \deg(v^{q_k}, G^{q_i})$. Since $v^{q_k} \notin \{i, j, k, i^{q_j^{-1}q_k}\}$, and since $G^{q_i} \oplus H = \{[i, k], [i, j]\}$, we conclude that $\deg(v^{q_k}, G^{q_k}) = \deg(v^{q_k}, H)$. If $v = k$, we also have $\deg(k, G) = \beta + 1 = \deg(k^{q_k}, H)$. The degrees of $\{j, j^{q_k^{-1}}, i^{q_j^{-1}}\}$ in G are $\{\alpha-1, \alpha-1, \alpha\}$. The degrees of v^{q_k} in H are $\{\alpha-1, \alpha, \alpha\}$, so that G and H have different degree sequences, a contradiction. We conclude that $|C_j| = 2$, so that $i^{q_j^{-1}} = j^{q_k^{-1}}$, and $G' - G = \{[j, i^{q_j^{-1}}]\}$.

We then find that q_j maps $[k, i^{q_j^{-1}}] \in G - G'$ to $[i, k] \in H - G^{q_i}$, and $[j, i^{q_j^{-1}}] \in G' - G$ is mapped to $[i, j] \in G^{q_i} - H$. Therefore $(G \oplus G')^{q_j} = H \oplus G^{q_i}$. Since $(G')^{q_j} = G^{q_i}$, it follows that $G^{q_j} = H$, as required.

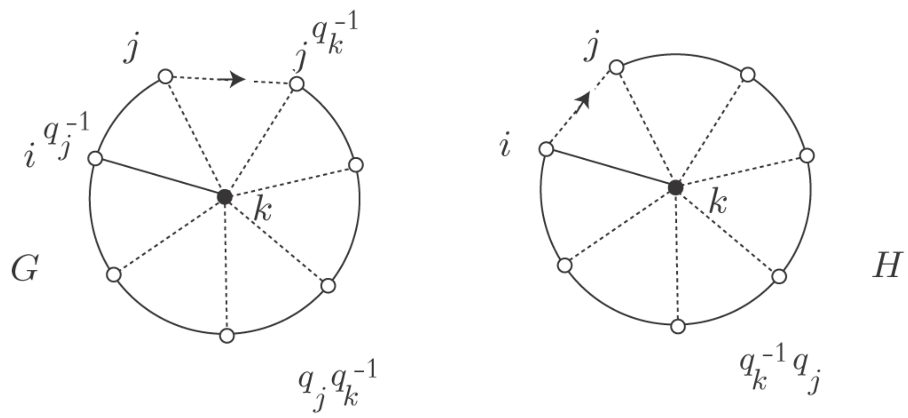

Figure 12, $G \oplus G' = \{[j, j^{q_k^{-1}}], [k, i^{q_j^{-1}}]\}$

Case 2. $i^{q_j^{-1}} = i^{q_k^{-1}}$

In this section we assume that $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 3$ and that $i^{q_j^{-1}} = i^{q_k^{-1}}$. Notice that $i^{q_j^{-1}}$ is a fixed point of $q_j q_k^{-1}$.

4.8 Lemma. *$G \oplus G'$ consists of pairs of the forms $[i^{q_j^{-1}}, x]$ or $[j, k]$.*
Proof. As in Lemma 3.3, we find that if $[x, y] \in G \oplus G'$, then $[x, y]^{p_j p_i^{-1}}$ and $[x, y]^{p_k p_i^{-1}}$ do not exist. Hence $\{x, y\} \cap \{j, i^{q_j^{-1}}\} \neq \emptyset$ and $\{x, y\} \cap \{k, i^{q_k^{-1}}\} \neq \emptyset$. Since $i^{q_j^{-1}} = i^{q_k^{-1}}$, this reduces to either $i^{q_j^{-1}} \in \{x, y\}$, or $[x, y] = [j, k]$.

4.9 Theorem. *If $i^{q_j^{-1}} = i^{q_k^{-1}}$, and $q_j q_k^{-1}$ is of type 2, then $G^{q_j} = H$.*
Proof. Let X be the set of vertices x such that $[i^{q_j^{-1}}, x] \in G \oplus G'$. Let $[i^{q_j^{-1}}, X]$ denote the set of all pairs $[i^{q_j^{-1}}, x]$ such that $x \in X$. By Lemma 4.8, $(G \oplus G') - [j, k] = [i^{q_j^{-1}}, X]$. Given $x \in X$, if $x^{p_j p_k^{-1}}$ exists, then $x^{p_j p_k^{-1}} \in X$, since $p_j p_k^{-1}$ is a partial automorphism of G and G'. If $x^{p_j p_k^{-1}}$ does not exist, then since $q_j q_k^{-1}$ is of type 2, we can apply $p_k p_j^{-1}$ a number of times to obtain $x^{p_j p_k^{-1}} \in X$. Thus $X^{q_j q_k^{-1}} = X$. By Lemma 3.3, $H \oplus G^{q_i} = [i, X^{q_j}]$, since

$X^{q_j} = X^{q_k}$. Since $|G \oplus G'|$ and $|H \oplus G^{q_i}|$ are even we conclude that $G \oplus G' = [i^{q_j^{-1}}, X]$, so that $(G \oplus G')^{q_j} = G^{q_j} \oplus G^{q_i} = [i, X^{q_j}] = G^{q_i} \oplus H$ so that $G^{q_j} = H$.

4.10 Theorem. *If $i^{q_j^{-1}} = i^{q_k^{-1}}$, and $q_j q_k^{-1}$ is of type 1, then either $G \cong H$ or else $G^{q_j} \oplus H = \{[i,j],[j,k^{q_j}]\}$*

Proof. Let C denote the cycle of $q_j q_k^{-1}$ containing j and k. Refer to Figure 13. As in Theorem 4.9, let X denote the set of vertices x such that $[i^{q_j^{-1}}, x] \in G \oplus G'$. By Lemma 4.8, $(G \oplus G') - [j,k] = [i^{q_j^{-1}}, X]$. Let $X' = X - C$. Then $(X')^{q_j q_k^{-1}} = X'$, since $p_j p_k^{-1}$ is a partial automorphism of both G and G', and C is a cycle of $q_j q_k^{-1}$. If $X^{q_j q_k^{-1}} = X$, we again have $G^{q_j} = H$, as in Theorem 4.9. We divide the cycle C into two subsets: P_1, the vertices on the path from k to j; and P_2, the vertices on the path from $j^{q_k^{-1}}$ to $k^{q_j^{-1}}$. Then $P_1 \cup P_2 = C$. If $x \in X \cap P_\ell$, where $\ell = 1$ or 2, then we can apply either $p_j p_k^{-1}$ or its inverse to x as many times as needed to obtain $P_\ell \subseteq X$. Consequently we can assume that $X \cap C$ is either P_1 or P_2.

Suppose first that $X \cap C = P_1$. By Lemma 3.3, $G^{q_i} \oplus H = [i, (X-j)^{p_j}] \cup [i, (X-k)^{p_k}]$. Let $m = |P_1|$. Then $|G^{q_i} \oplus H| = |X'| + m - 1$, whereas $|[i^{q_j^{-1}}, X]| = |X'| + m$. Since $|G^{q_i} \oplus H|$ and $|G \oplus G'|$ are both even, we conclude that $[j,k] \in G \oplus G'$. We then have $(G \oplus G')^{q_j} = G^{q_j} \oplus G^{q_i} = [i^{q_j^{-1}}, X]^{q_j} \cup \{[j,k]^{q_j}\} = [i, X^{q_j}] \cup \{[j, k^{q_j}]\}$. Comparing this with the expression for $G^{q_i} \oplus H$ gives $G^{q_j} \oplus H = \{[i,j],[j,k^{q_j}]\}$.

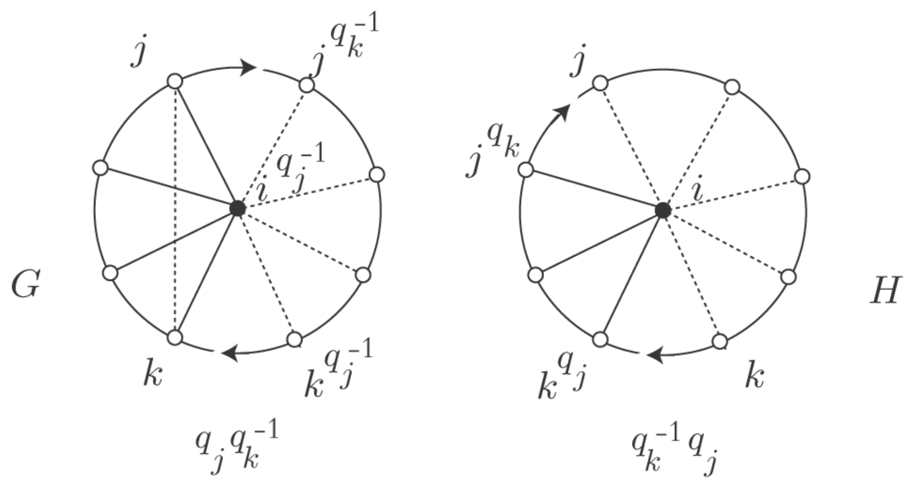

Figure 13, $i^{q_j^{-1}} = i^{q_k^{-1}}$

Suppose now that $X \cap C = P_2$. Let $\ell = |P_2|$. We find that $G^{q_i} \oplus H = [i, X^{p_j}] \cup [i, X^{p_k}]$, since $j, k \notin X$. Therefore $|G^{q_i} \oplus H| = |X'| + \ell + 1$, whereas $|[i^{q_j^{-1}}, X]| = |X'| + \ell$. It follows that $[j,k] \in G \oplus G'$. As above, we obtain $G^{q_j} \oplus H = \{[i,j],[j,k^{q_j}]\}$.

We have been unable to prove that $G^{q_j} = H$ in this case. If we attempt to use the degree sequence as in Theorem 3.5, we can proceed as follows. Consider the case when $X \cap C = P_1$, and assume that $[i^{q_j^{-1}}, P_1] \subseteq G - G'$. Then $[i^{q_j^{-1}}, j] \in G - G'$ so that $[i,j] \in G^{q_j}$. Since $G^{q_j} \oplus H = \{[i,j],[j,k^{q_j}]\}$, we have $[i,j] \in G^{q_j} - H$ and $[j,k^{q_j}] \in H - G^{q_j}$. Therefore

$[j,k] \in G' - G$. Let $m_1 = |(G' - G) \cap [i^{q_j^{-1}}, X']|$ and $m_2 = |(G - G') \cap [i^{q_j^{-1}}, X']|$. Then $|G' - G| = m_1 + 1$ and $|G - G'| = m + m_2$, so that $m_1 + 1 = m + m_2$.

In G', the vertices of C all have the same degree, call it α. Let $\beta = \deg(i^{q_j^{-1}}, G')$. Then $\deg(i^{q_j^{-1}}, G) = \beta + m + m_2 - m_1 = \beta + 1$. Also, $\deg(j, G) = \deg(k, G) = \alpha$. The other vertices of P_1 have degree $\alpha + 1$ in G. We use $G^{q_j} - H = \{[i,j]\}$ and $H - G^{q_j} = \{[j, k^{q_j}]\}$ to obtain the degrees in H. We have $\beta + 1 = \deg(i^{q_j^{-1}}, G) = \deg(i, G^{q_j}) = \deg(i, H) + 1$, so that $\deg(i, H) = \beta$. Also, $\alpha = \deg(k, G) = \deg(k^{q_j}, G^{q_j}) = \deg(k^{q_j}, H) - 1$, so that $\deg(k^{q_j}, H) = \alpha + 1$. If $v \notin \{i, k^{q_j}\}$, then $\deg(v, G^{q_j}) = \deg(v, H)$. The only conclusion drawn is that $\alpha = \beta$.

Case 3. $j = i^{q_k^{-1}}$ **and** $k = i^{q_j^{-1}}$

In this section we assume that $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 2$ and that $j = i^{q_k^{-1}}$ and $k = i^{q_j^{-1}}$. These conditions weaken the implications of the relation $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$, and do not seem sufficient to settle this case. We have only the following simple lemma and observations.

4.11 Lemma. *$G \oplus G'$ consists of pairs of the forms $[j,x]$ or $[k,x]$.*
Proof. As in Lemma 3.3, we find that if $[x,y] \in G \oplus G'$, then $[x,y]^{p_j p_i^{-1}}$ and $[x,y]^{p_k p_i^{-1}}$ do not exist. Hence $\{x,y\} \cap \{j, i^{q_j^{-1}}\} \neq \emptyset$ and $\{x,y\} \cap \{k, i^{q_k^{-1}}\} \neq \emptyset$. Since $i^{q_j^{-1}} = k$ and $i^{q_k^{-1}} = j$, this reduces to either $j \in \{x,y\}$, or $k \in \{x,y\}$.

Notice that $k^{q_j q_k^{-1}} = (i^{q_j^{-1}})^{q_j q_k^{-1}} = i^{q_k^{-1}} = j$. Therefore $q_j q_k^{-1}$ is of type 1. Suppose that $[k,x] \in G \oplus G'$, for some x. If $[k,x]^{p_j p_k^{-1}}$ exists, then $[j, x^{q_j q_k^{-1}}]$ is also in $G \oplus G'$, since $p_j p_k^{-1}$ is a partial automorphism of G, and an automorphism of G'. If it does not exist, then x is either j or $k^{q_j^{-1}}$. Thus, the possible difference pairs are $[k, k^{q_j^{-1}}]$, $[j,k]$, and $[j, j^{q_k^{-1}}]$ (by symmetry); and pairs of the form $[k,x]$, $[j, x^{q_j q_k^{-1}}]$.

5. Conclusion

The algebraic relations $G^{q_i q_j^{-1}} = G^{q_j q_i^{-1}}$ or $G^{q_i q_j^{-1}} = G^{q_i q_k^{-1}}$ are sufficient to force the reconstructibility of G in many cases. The second relation implies that one of G^{q_i}, G^{q_j}, or G^{q_k} always equals H when $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 4$. When $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 3$, the results are less conclusive; G^{q_j} and H can sometimes differ in up to two edges, that is $|G^{q_j} \oplus H| \leq 2$. When $|\{j, k, i^{q_j^{-1}}, i^{q_k^{-1}}\}| = 2$, the conditions are not strong enough to force $G \cong H$. Other algebraic relations for which similar results likely hold are $G^{q_i q_j^{-1}} = G^{q_k q_i^{-1}}$ and $G^{q_i q_j^{-1}} = G^{q_k q_\ell^{-1}}$, where i, j, k and ℓ are distinct.

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