Fano Quads in Projective Planes

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Abstract

There are two types of quadrangles in a projective plane, Fano quadrangles, and non-Fano quadrangles. The number of quadrangles in some small projective planes is counted according to type, and an interesting configuration in the Hughes plane is displayed.

1 Fano Quads

Let P be a finite projective plane of order $n \geq 2$, so that every point lies on $n+1$ lines and every line contains $n+1$ points, there being n^2+n+1 points in total and n^2+n+1 lines in total. A *quadrangle*, which we abbreviate to *quad*, is any set of 4 points, no 3 collinear, which then determines 6 lines in pairs. If $\{A, B, C, D\}$ is a quad, and AB, AC, AD, BC, BD, CD, are the 6 lines determined, then the diagonal triangle E, F, G, is the set of 3 points determined by the intersections of opposite lines, namely $E = AB \wedge CD$, $F = AC \wedge BD$, and $G = AD \wedge BC$. In case the 3 points E, F, G, are collinear, we say that $\{A, B, C, D\}$ is a *Fano* quad, because the 7 points $\{A, B, C, D, E, F, G\}$ and the seven lines described above form a Fano plane, that is, a projective plane of order 2. Otherwise we say that $\{A, B, C, D\}$ is a *non-Fano* quad.

So a plane P which contains a Fano quad contains a subplane of order 2. The Fano plane can only be coordinatized by a field of characteristic 2.

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Therefore, if P is coordinatized by a field, P will contain no Fano quads unless its field has characteristic 2. Conversely, if its field has characteristic 2, then every quad of P will be a Fano quad. In a field plane, all quads are either Fano quads, or else all are non-Fano quads (cf [1]).

We can count the quads in a plane of order n. Let $\{A, B, C, D\}$ be a quad. There are n^2+n+1 choices for A. Having chosen A, there are n^2+n choices for B. The line AB contains $n+1$ points, and C can not be one of them. So there are n^2 choices for C. We now have 3 lines, AB, AC, BC, containing together $3n$ points. So there are n^2-2n+1 choices for D. The total number of quads in P is therefore

$$Q = (n^2+n+1)(n^2+n)(n^2)(n^2-2n+1)/24 = n^3(n^3-1)(n^2-1)/24$$

as each quad has been counted 24 times.

n	2	3	4	5	7	8	9
Q	7	234	2520	15500	234612	686784	1769040

2 The Plane of Order 4

The projective plane of order 4 is completely determined, up to isomorphism, by any of its quads. Let $PP(4)$ denote any projective plane of order 4. Let A, B, C, D be a quad. Taking $E = AB \wedge CD$, $F = AC \wedge BD$, and $G = AD \wedge BC$, we have 7 points and 6 partial lines in $PP(4)$, namely $ABE, CDE, ACF, BDF, ADG, BCG$. Any 2 of these intersect in a point. Therefore they have no more common points. Since each line of $PP(4)$ contains 5 points, there are 2 more points per line, giving exactly 12 more points to complete these lines. We now have 19 points out of 21. If $\{A, B, C, D\}$ were a non-Fano quad, then EF, EG, FG would form 3 additional partial lines of $PP(4)$, each requiring 3 more points to complete. Since EF, EG, FG must intersect each of the above 6 lines in exactly one point, we find that at least one more point is required for each of EF, EG, FG, giving 22 points in $PP(4)$, a contradiction.

We conclude that all quads in $PP(4)$ are Fano quads. Accordingly, EFG forms an additional partial line. The 7 partial lines require 14 additional points to complete. Call them $\{a, b, c, d, e, f, g, h, i, j, k, l, m, n\}$. This gives all 21 points in total. Without loss of generality, the 2 remaining lines containing A can be taken to be $Acgkm$ and $Adhln$. The 2 remaining lines containing B then become $Bdeim$ and $Bcfjn$, without loss of generality. The remaining 2 lines containing C and D can then be chosen as in Figure 1. The lines containing E, F and G are then completely forced.

FANO

A	C	A	B	A	B	E	A	A	B	B	C	C	D	D	E	E	F	F	G	G
B	D	C	D	D	C	F	c	d	d	c	a	b	b	a	e	f	a	b	a	b
E	E	F	F	G	G	G	g	h	e	f	h	g	f	e	g	h	c	d	d	c
a	c	e	g	i	k	m	k	l	i	j	j	i	l	k	j	i	i	j	f	e
b	d	f	h	j	l	n	m	n	m	n	m	n	m	n	l	k	l	k	g	h

Figure 1: A Fano subplane in $PP(4)$

Combinatorial argument for the uniqueness of $PP(4)$ is given in reference [2].

3 Fano Residuals

Let H be a plane of order n containing a subplane Q of order m. The *residual* of Q is $\overline{Q}$, obtained from H by deleting certain points and lines. The m^2+m+1 lines of Q each contain an additional $n+1-(m+1)=n-m$ points, giving a set U of $(n-m)(m^2+m+1)$ points. See Figure 2 where this is illustrated for $n=9$ and $m=2$. The remaining $n^2+n+1-(n-m)(m^2+m+1)-(m^2+m+1)=(n-m)(n-m^2)$ points which do not occur on the lines of Q are called residual points. Call the set of residual points R. Since $|R|\geq 0$, we have $(n-m)(n-m^2)\geq 0$. This produces the well-known result that a plane of order n can only have a subplane of order $m\leq\sqrt{n}$.

Each of the m^2+m+1 points of Q occurs in an additional $n-m$ lines in H, giving an additional $(n-m)(m^2+m+1)$ lines. Each point of U must occur in m^2 of these lines, accounting for $m^2(n-m)(m^2+m+1)$ points out of $n(n-m)(m^2+m+1)$ total points on these lines. The remaining $(m^2+m+1)(n-m)(n-m^2)$ points on these lines must be filled by points of R. This means that each point of R occurs $(n-m)(n-m^2)(m^2+m+1)/(n-m)(n-m^2)=(m^2+m+1)$ times each on these lines. Each residual point must therefore occur in an additional $(n+1)-(m^2+m+1)$ lines. This leaves a configuration of $(n-m)(n-m^2)$ residual lines, each containing $n-m-m^2$ residual points, and (m^2+m+1) points of U.

The residual of Q is the configuration induced by the $(n-m)(n-m^2)$ residual points and $(n-m)(n-m^2)$ residual lines. Each residual point occurs $n-m-m^2$ times on these lines. In case $m=2$, we call $\overline{Q}$ a Fano residual.

Remark: It is well-known that $n=m^2$ is possible for a subplane of order

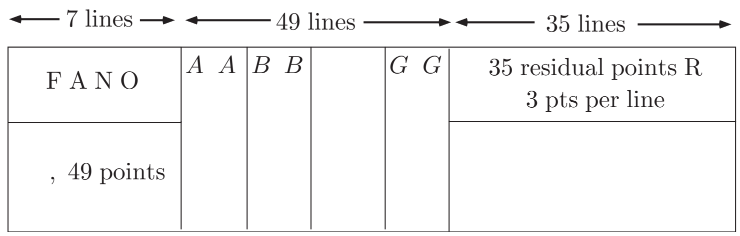

Figure 2: A Fano residual in a plane of order 9

m in a plane of order n. However $n = m^2 + 1$ is not possible, since the residual would then have $m^2 - m + 1$ points each occuring $1 - m$ times on $m^2 - m + 1$ lines. This would require $m \leq 1$, which is impossible.

For example, if $n = 4$ and $m = 2$, the residual has $(n-2)(n-4) = 0$ points. If $n = 8$ and $m = 2$, it has 24 lines and 24 points, each occurring on 2 lines. If $n = 9$ and $m = 2$, the residual has 35 points and 35 lines, each line containing 3 points.

4 The Plane of Order 8

The plane of order 8 has 73 points in total. The residual of a Fano quad contains 24 points and 24 lines, each conaining 2 points. In the field plane, these pairs in the residual define 8 triangles [3].

5 The Planes of Order 9

There are 4 planes of order 9; they are the field plane, the Hughes plane, the Hall plane, and the dual of the Hall plane. The field plane and the Hughes plane are both self-dual. The field plane contains no Fano quads. The other planes contain both Fano quads and non-Fano quads. Let us start with the Hughes plane H, which has 91 points and lines. Referring to the above with $n = 9$ shows that a subplane must have order $m \leq 3$, and it is known that H contains a subplane isomorphic to $PP(3)$ (cf [1]). Let Q be a Fano quad in H; then Q may intersect the subplane $PP(3)$. Note that $Q \cap PP(3)$ can not contain 4 points, since $PP(3)$ contains no Fano quad. Therefore $Q \cap PP(3)$ contains 0, 1, 2, or 3 points. We have counted all the quads in H, Fano and non-Fano, and tabulated them according to $|Q \cap PP(3)|$. The results are as follows.

1	1	1	2	2	2	3	3	3	4	4	5	5	5	6	6	6	7	8	9	10	10	11	12	13	14	21
4	7	27	8	9	13	9	11	15	7	17	12	19	23	8	10	15	17	14	13	12	20	16	17	20	15	22
14	11	28	25	19	18	26	27	22	24	28	18	22	25	21	19	20	27	28	16	16	26	21	24	23	18	23

Figure 3: The 28 triples of the $(3, 3)$-configuration

$\lvert Q \cap PP(3) \rvert$	Fano quads	non-Fano quads
0	129168	816426
1	84240	566280
2	16848	141804
3	5616	8424
4	0	234
total	235872	1533168

We consider the possible Fano residuals in the Hughes plane. There are only 3 possible residuals, and they are all self-dual. They have automorphism groups of order 2, 6, and 1008, acting on the points. The residual with a group of order 1008 is particularly interesting. It consists of a Fano configuration, $PP(2)$, and a self-dual configuration R with 28 points and 28 lines. The Fano configuration has a group of order 168, and R has a group of order 6. The triples of the configuration R are listed in Figure 3, and two different drawings of R as a projective $(3, 3)$-configuration are shown in Figures 4 and 5. Several of the lines are drawn as circles, and several are drawn as arcs. Each line contains three points. The outer circle is drawn with each of its three points appearing twice, as this simplifies the drawing. The lines through point 19 are shown separately to avoid cluttering the diagram. Point 19 is fixed by all automorphisms. The outer circle and point 19 are polar opposites.

The group of this configuration is generated by the two permutations

(1,7,4)(2,22,10)(3,12,8)(5,6,9)(11,24,14)(13,23,20)(15,16,25)(17,28,27)(18,21,26),
(1,4)(2,6)(3,12)(5,22)(9,10)(11,24)(13,20)(15,18)(16,26)(17,27)(21,25),

as can be seen from the drawing.

Definition 5.1 *Given two subplanes Q_1 and Q_2 of a plane H, we say that Q_1 and Q_2 are disjoint if they have disjoint sets of points and lines, and if furthermore, the intersection of any line of Q_1 with any line of Q_2 (as lines in H), is not a point of either Q_1 or Q_2.*

We summarize the above paragraphs in a lemma.

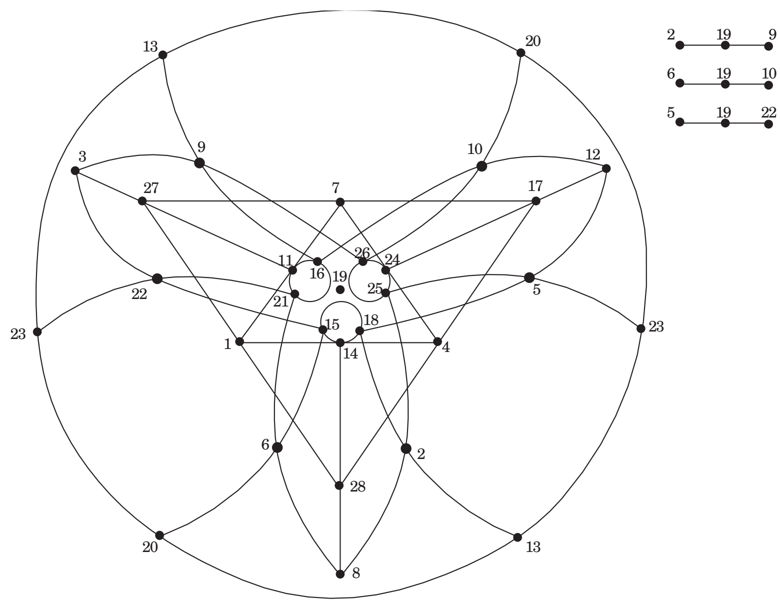

Figure 4: The $(3, 3)$-configuration R on 28 points

Lemma 5.1 *The Hughes plane contains pairs of disjoint Fano subplanes. The 28 points not contained in any line of either Fano subplane induce a self-dual* $(3, 3)$*-configuration isomorphic to* R*.*

An additional drawing of this configuration is shown in Figure 5.

In the Hall plane there are 362,880 Fano quads and 1,406,160 non-Fano quads. The Fano residuals are all isomorphic. Each residual is a non-self-dual $(3, 3)$-configuration on 35 points, with an automorphism group that has order 6.

Lemma 5.2 *Let* H *be a projective plane of order 9, containing a Fano subplane* Q_1*. Let* R_1 *be the residual points of* Q_1*. If* H *contains a Fano subplane* Q_2 *disjoint from* Q_1*, then the points of* Q_2 *belong to* R_1*.*

Proof Let H contain a disjoint Fano subplane Q_2. Refer to Figure 2. By 5.1, the lines of Q_2 must be lines of $\overline{Q}_1$, the residual of Q_1. If Q_2 contained a point $X \notin R_1$, then the lines of Q_2 containing X would all intersect a line of Q_1 in X, contrary to 5.1. It follows that points of Q_2 belong to R_1.

Corollary 5.3 *The Hall plane and its dual do not contain two disjoint Fano subplanes.*

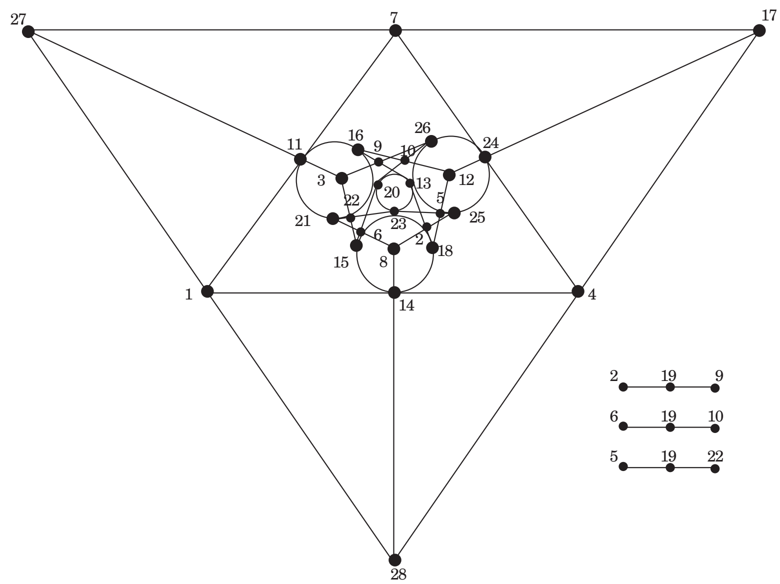

Figure 5: The $(3,3)$-configuration R on 28 points

Proof Let H denote the Hall plane or its dual. If there were disjoint Fano subplanes Q_1 and Q_2, then by 5.3, the points of Q_2 must all be in R_1. Each line of Q_2 must contain 3 points of R_1. Since each line of $\overline{Q}_1$ contains 3 points of R_1, a Fano subplane Q_2 is easily recognized in a residual – the incidence graph of $\overline{Q}_1$ would contain a connected component isomorphic to the incidence graph of the Fano plane. But this does not occur in H or its dual. This completes the proof.

Corollary 5.4 *The Hughes plane contains a clique covering of K_{28} with 49 K_4's and 28 K_3's.*

Proof Let H denote the Hughes plane. Let Q_1 and Q_2 be disjoint Fano subplanes, as in 5.3. There are 35 residual points R_1 with respect to Q_1 (see Figure 2). Of these 35 points, 7 are points of Q_2. This leaves 28 points in $R = R_1 \cap R_2$. The 35 lines of the residual $\overline{Q}_1$ each contain 3 points of R. Of these, 7 belong to Q_1. There remain 28 triples of points of R. This defines the 28 K_3's. The lines of H containing 1 point of Q_1 must also contain exactly 1 point of Q_2. They will also each contain exactly 4 points of R. There are 49 such lines. This defines the 49 K_4's. Together the K_3's and K_4's contain every pair of K_{28} exactly once.

References

[1] J.W.P. Hirschfeld, *Projective Geometries over Finite Fields*, Oxford, Clarendon Press, 1998.

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