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Embedding Graphs Containing K_5-Subdivisions

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Abstract

Given a non-planar graph G with a subdivision of K_5 as a subgraph, we can either transform the K_5-subdivision into a $K_{3,3}$-subdivision if it is possible, or else we obtain a partition of the vertices of $G\backslash K_5$ into equivalence classes. As a result, we can reduce a projective planarity or toroidality algorithm to a small constant number of simple planarity checks [6] or to a $K_{3,3}$-subdivision in the graph G. It significantly simplifies algorithms presented in [7], [10] and [12]. We then need to consider only the embeddings on the given surface of a $K_{3,3}$-subdivision, which are much less numerous than those of K_5.

1. Introduction

We use basic graph-theoretic terminology from Bondy and Murty [1] and Diestel [2]. Let G be a 2-connected, undirected, simple graph. We are interested in a practical efficient algorithm to decide whether G can be embedded in the projective plane or torus.

Known algorithms in [7], [10] and [12] begin with a Kuratowski subgraph K_5 or $K_{3,3}$ in G, and try to extend an embedding of K_5 or $K_{3,3}$ to an embedding of G in the projective plane or torus. Here we show that non-planar graphs which do not contain a $K_{3,3}$-subdivision are much easier to embed than using the methods of [7], [10] or [12]. As there are many labelled embeddings of K_5 on the projective plane or torus, this eliminates many of the cases which must be considered in the before mentioned algorithms. However there are only two non-isomorphic embeddings of $K_{3,3}$ on the torus, and just one in the projective plane. The result is a considerable simplification of the existing algorithms.

One needs a combinatorial description of a graph embedded on a surface. Such a description is provided by a *rotation system* (cf. [5]) of the

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graph, which is a set of cyclically ordered adjacency lists of its vertices. For a non-orientable surface the rotation system also includes a *signature* for every edge. The signature of an edge is $+1$ or -1. It is negative when the edge goes "over the boundary" and positive otherwise. For a more detailed description see [5].

The following theorem is well known.

Kuratowski's Theorem. [9] *A graph G is non-planar if and only if it contains a subdivision of $K_{3,3}$ or K_5.*

Hopcroft and Tarjan [6] developed an efficient practical linear time algorithm to check if a graph G is planar or not. If G is planar, then its planar rotation system can always be transformed into a 2-cell toroidal or projective planar rotation system. We developed several methods to do this transformation [4] and implemented them in the software *Groups&Graphs* [8].

In general, a planarity testing algorithm can be modified so that in case of a non-planar graph G it will return a subdivision of K_5 or $K_{3,3}$ in G. We assume that G is a non-planar, 2-connected graph with no vertices of degree 2.

We describe the structure of non-planar graphs with no $K_{3,3}$-subdivision of a special type. These graphs have a partition of the vertices and edges into equivalence classes. We can then proceed by recursion on the subgraphs generated by the equivalence classes in G.

2. Structural Results

Let G be a non-planar graph. Following Diestel [2], we denote by TK_5 a K_5-subdivision in G. We call the vertices of degree 4 *corners* of TK_5 and the vertices of degree 2 *inner vertices* of TK_5. A path between two corners with all other vertices inner vertices is called a *side* of the K_5-subdivision. Notice that two sides of a K_5-subdivision can have at most one common corner and no common inner vertices. A side having a common corner with another side is called *adjacent* to that side. Two sides having no common corner are called *non-adjacent*.

A path P in G with one endpoint an inner vertex of TK_5, the other endpoint on a different side of TK_5, and all other vertices and edges in $G \backslash TK_5$ is called a *short cut* of the K_5-subdivision. A vertex $u \in G \backslash TK_5$ is called a 3-*corner vertex* with respect to TK_5 if $G \backslash TK_5$ contains internally disjoint paths from u to at least three corners of the K_5-subdivision (see Fig. 4).

We begin by proving some basic structural results for graphs containing a TK_5. Similar structural results have been proved previously by M. Fellows and P. Kaschube [3]. We note that their proof of Theorem 1 [3] is missing the case indicated by Fig. 1 of Proposition 2.1.

2.1 Proposition. [3] *A non-planar graph G with a K_5-subdivision TK_5 for which there is either a short cut or a 3-corner vertex contains a $K_{3,3}$-subdivision.*

Proof. To prove the proposition, we exhibit a $K_{3,3}$-subdivision in G. In the following diagrams the bipartition of $K_{3,3}$ is indicated by black and white vertices. Vertices which are not part of $K_{3,3}$ are shaded grey.

The following cases are possible.

Case 1. Both endpoints of a short cut P are inner vertices of TK_5 and the corresponding two sides are non-adjacent. Fig. 1 shows a $K_{3,3}$-subdivision in G.

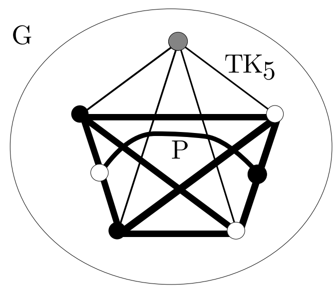

Fig. 1, $K_{3,3}$ created by short cut P

Case 2. Both endpoints of a short cut P are inner vertices of TK_5 and the corresponding two sides are adjacent. Fig. 2 shows a $K_{3,3}$-subdivision in G.

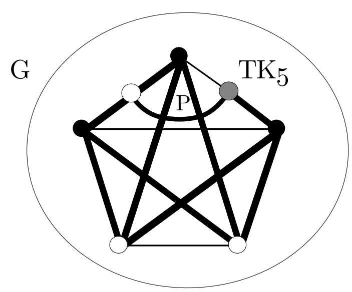

Fig. 2, $K_{3,3}$ created by short cut P

Case 3. One of the endpoints of a short cut P is a corner of TK_5. Fig. 3 shows a $K_{3,3}$-subdivision in G.

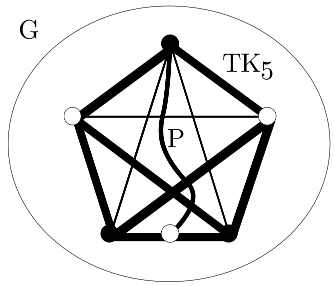

Fig. 3, $K_{3,3}$ created by short cut P

Now suppose there is a 3-corner vertex $u \in G \backslash TK_5$. Then Fig. 4. shows a $K_{3,3}$-subdivision in G.

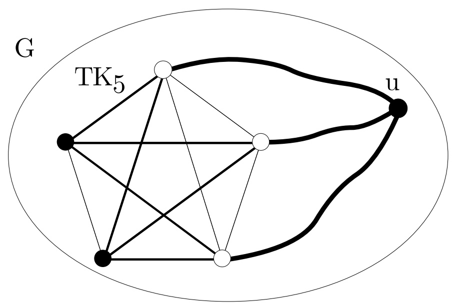

Fig. 4, $K_{3,3}$ created by 3-corner vertex u

Thus any short cut or 3-corner vertex of TK_5 in G gives a $K_{3,3}$-subdivision. The proposition is proved.

Let G be a graph having no 3-corner vertex and no short cut of TK_5. Let K be the set of corners of TK_5. Consider the set of connected components of $G \backslash K$. Let C be any connected component of $G \backslash K$.

2.2 Proposition. [3] *For a graph G with TK_5 and no short cut or 3-corner vertex of TK_5, a connected component C of $G \backslash K$ contains inner vertices of at most one side of TK_5. Moreover vertices of C are adjacent in G to exactly two corners of TK_5.*

Proof. Suppose a connected component C contains inner vertices of two different sides of TK_5. Then clearly C contains a short cut of TK_5 in G, a contradiction.

Suppose C has vertices adjacent in G to at least 3 different corners of TK_5. Then it is not difficult to show that C contains either a short cut or a 3-corner vertex of TK_5, a contradiction. Therefore, vertices of C are adjacent in G to at most 2 corners of TK_5. Since G is 2-connected, we have exactly two such corners.

Given a graph G without a short cut or a 3-corner vertex of TK_5 we define a *side component* of TK_5 as a subgraph in G induced by a pair of corners a and b of TK_5 and all connected components of $G \backslash K$ adjacent to a and b.

2.3 Corollary. *Two side components of TK_5 in G have at most one vertex in common. The common vertex is the corner of intersection of two corresponding sides of TK_5.*

Proof. Any pair of corners of TK_5 define a side. Since G is 2-connected, by Proposition 2.2, we can associate every connected component of $G \backslash K$ with a unique side of TK_5. This gives a partition of vertices of $G \backslash K$ into side components of TK_5.

Notice that side components, however, can contain a $K_{3,3}$-subdivision. Thus, given a graph G with a K_5-subdivision TK_5, either we can find a short cut or a 3-corner vertex of TK_5 in the graph, or else we can partition the vertices and edges of $G \backslash TK_5$ into equivalence classes according to the corresponding side components of TK_5 in G.

Every side component H of TK_5 contains exactly two corners a and b corresponding to a side of TK_5. If edge ab between the corners is not in H, we can add it to H to obtain $H+ab$. Otherwise $H+ab = H$. We call ab the *corner edge* of $H+ab$ and $H+ab$ an *augmented side component* of TK_5. We use the following general lemma for side components of a K_5-subdivision in the embedding algorithms. By Lemma 2.4 we can add the corner edge into every side component H to test easily if there exists an embedding of H with both corners on the outer face and to find such an embedding.

2.4 Lemma. *There is a planar embedding of a graph G with two vertices u and v on the outer face if and only if there exists a planar embedding of the graph $G + uv$.*

Proof. It can be seen by drawing any planar embedding on the sphere that any face of a planar embedding can be considered as an outer face. Now if there exists an embedding of graph G on the sphere with both vertices u and v on the same face, then we can just add the edge between them into the face. Otherwise for any embedding of G on the sphere the edge cannot be added into the planar embedding. Hence $G + uv$ is not planar.

3. Algorithm for the Projective Plane

Let G be a non-planar graph with a K_5-subdivision TK_5. We can use Propositions 2.1 and 2.2 either to determine if G is projective planar or toroidal or to find a $K_{3,3}$-subdivision in it. In this section, we present a linear time practical algorithm to check if a non-planar graph G is projective planar or contains a $K_{3,3}$-subdivision. We begin with a characterization of projective planarity for graphs with a K_5-subdivision.

3.1 Theorem. *A graph G with a K_5-subdivision TK_5 and no short cut or 3-corner vertex of TK_5 is projective planar if and only if all the augmented side components of TK_5 are planar graphs.*

Proof. By Corollary 2.3, all the vertices and edges of $G \backslash TK_5$ are partitioned into side components.

Sufficient conditions. Take any embedding of TK_5 on the projective plane (see Fig. 5). For each side of TK_5, make a planar embedding of its side component with both corners on the outer face. By Lemma 2.4 there exists such an embedding of a side component if and only if the augmented side component is a planar graph. By Corollary 2.3 we can embed every side component independently.

Necessary conditions. Fig. 5 shows the two possible non-isomorphic embeddings of TK_5 on the projective plane (see [10] and [12] for details).

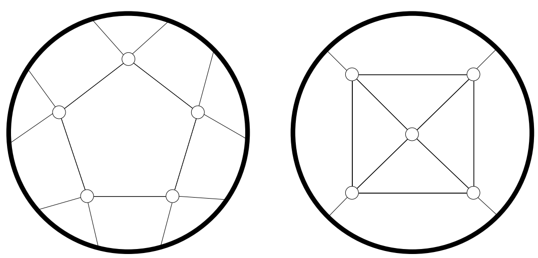

Fig. 5, The two embeddings of K_5 on the projective plane.

The sides of TK_5 must create one of these embeddings, which divides the projective plane into faces. Each vertex of TK_5 appears at most once on the boundary of any face, and every side of TK_5 is incident on exactly 2 faces. Call these faces F_1 and F_2, and let K be the set of corners of TK_5. For some sides, it is possible that the two corners a and b also appear on the boundary of a third face, as non-consecutive vertices. But since G has no short cut or 3-corner vertex of TK_5, every connected component C of $G \backslash K$, adjacent to a and b and embedded in a third face can also be embedded in F_1 or F_2. This shows that it is always possible to embed every side component of TK_5 in an open disk contained in $F_1 \cup F_2$, i.e. every augmented side component must be planar. The theorem is proved.

Theorem 3.1 gives us a linear time practical algorithm for graphs with a K_5-subdivision.

Projective Plane Embedding Algorithm for Graphs with a K_5-Subdivision.

Input: *a graph G*

Output: *Either a projective planar rotation system of G, or a $K_{3,3}$-subdivision in G, or G is not projective planar*

1. Use a planarity checking algorithm (eg. [6]) to determine if G is planar. If G is planar then return its planar rotation system. If G is not planar and the planarity check returns a $K_{3,3}$-subdivision in G then return the $K_{3,3}$-subdivision in G.
2. G is not planar and the planarity check returned a K_5-subdivision TK_5 in G. Do a depth-first or breadth-first search to find either a short cut or a 3-corner vertex of TK_5 in G. If a short cut or a 3-corner vertex is found, then return a $K_{3,3}$-subdivision in G. If there is no short cut or 3-corner vertex, the depth-first or breadth-first search returns the side components of TK_5.
3. For each augmented side component H of TK_5 in G, check if H is planar. If all the augmented side components are planar, then return a projective planar rotation system of G. If there is a non-planar augmented side component of TK_5, then return G is not projective planar.

As every step in this algorithm has linear time complexity, the entire algorithm is also linear.

4. Algorithm for the Torus

Now consider the torus. Here we describe a linear time practical algorithm to check if a graph G is toroidal or contains a $K_{3,3}$-subdivision.

We begin with the 6 embeddings of K_5 on the torus, shown as $E_1, \ldots,$ E_6 in Fig. 6. Some embeddings have a face whose boundary contains a repeated vertex or repeated edge. Such a face is labelled F in the diagram. Vertices which are repeated on the boundary of F are shaded black. Repeated edges are drawn with thicker lines.

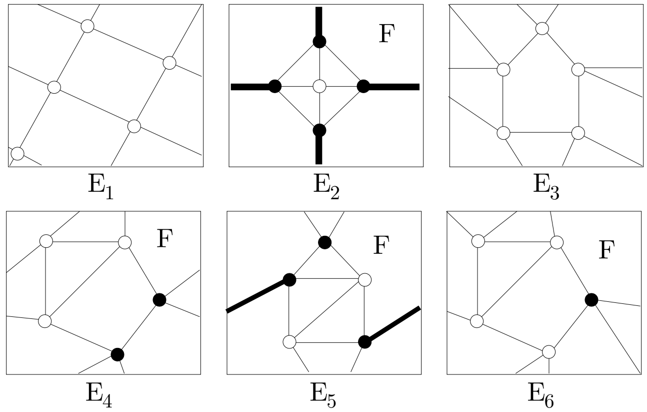

Fig. 6, Embeddings of K_5 on the torus.

Let G be a non-planar graph with a K_5-subdivision TK_5 and no short cut or 3-corner vertex of TK_5 in G. The following propositions and theorem provide a characterization of toroidality for such graphs.

4.1 Proposition. *If G is toroidal, then at most one augmented side component of TK_5 is non-planar.*

Proof. Let G be embedded on the torus. Consider the embeddings of K_5 on the torus $E_1, \ldots, E_6$ of Fig. 6. TK_5 must be embedded in one of these configurations. Let H be any side component with corners a and b. The vertices of H cannot be adjacent to any part of TK_5, except those vertices on the ab-side. We show that either $H + ab$ is planar, or else all other augmented side components are planar.

Case 1. TK_5 is embedded as E_1 or E_3 of Fig. 6.

E_1 and E_3 have the property that each vertex appears at most once on the boundary of any face. The side ab appears on the common boundary of 2 faces, say F_1 and F_2. Vertices a and b may also appear as non-consecutive vertices on the boundary of a third face. We proceed as in Theorem 3.1. Any portion of H embedded in a third face can be moved to F_1 or F_2, so that H can always be embedded in an open disk contained in $F_1 \cup F_2$, with a and b on the outer face of H. Lemma 2.4 implies that $H + ab$ is planar.

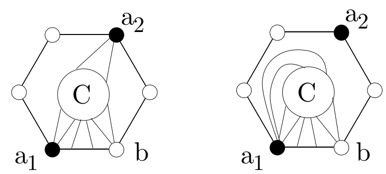

Fig. 7, A face with one repeated vertex.

Case 2. TK_5 is embedded as E_6 of Fig. 6.

The boundary of the face F contains one vertex repeated twice as in Fig. 7. Without loss of generality, we can assume that a is the repeated corner, and b is adjacent to a on the boundary of F. Otherwise $H + ab$ would be planar, as in Case 1. Let C be a part of H embedded in the interior of F. Let a_1 and a_2 be the two occurrences of a on the boundary of F, and let a_1 be adjacent to b on the facial foundary. The edges from a_2 to vertices $v \in C$ can be replaced by edges from a_1 to v, as indicated in the diagram. This gives a planar embedding of H with a and b on the outer face. Hence $H + ab$ is planar.

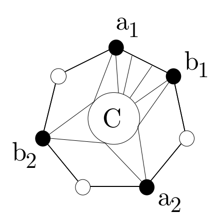

Fig. 8, A face with two repeated vertices.

Case 3. TK_5 is embedded as E_4 of Fig. 6.

The boundary of the face F of E_4 has 2 vertices repeated twice as in Fig. 8. Without loss of generality, we can take one of them to be a. Let its two occurrences on the facial boundary be a_1 and a_2. If b is not the other repeated corner, we can proceed as in Case 2 and $H + ab$ is planar. Hence, we can assume that b is also repeated. Let its two occurrences be b_1 and b_2, where b_1 is adjacent to a_1 on the facial boundary. Let C be the portion of H embedded inside F. Each of a_2 and b_2 must be adjacent to one or more vertices of C, or else we can proceed as in Case 2, and $H + ab$ is planar. Having embedded C as shown in Fig. 8, all faces of the embedding now have no repeated vertices on their boundaries. Consequently, all remaining augmented side components must be planar, as in Case 1. It follows that $H + ab$ is the only possible non-planar augmented side component.

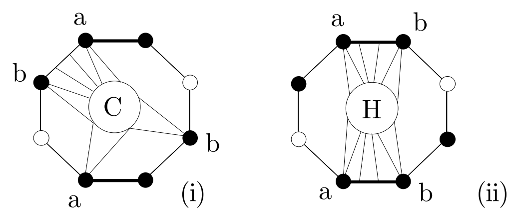

Fig. 9, A face with three repeated vertices.

Case 4. TK_5 is embedded as E_5 of Fig. 6.
The boundary of the face F of E_5 has an edge and another vertex repeated twice as in Fig. 9. If just one corner of the side ab is repeated twice on the boundary of F, then it is equivalent to Case 2 and $H+ab$ is planar. If both corners a and b are repeated twice on the boundary of F, but the side itself appears just once (Fig. 9(i)), we have a case similar to Case 3 and $H+ab$ is the only possible non-planar augmented side component in G. Suppose the entire side ab appears twice on the boundary of F, and H is embedded in F as in Fig. 9(ii). Then we find that after embedding H, any face of the embedding has at most one repeated corner as in case 2. Consequently, there can be at most one non-planar augmented side component $H+ab$.

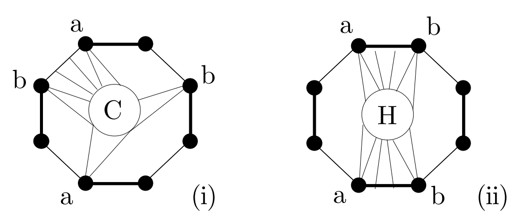

Fig. 10, A face with four repeated vertices.

Case 5. TK_5 is embedded as E_2 of Fig. 6.
The boundary of the face F of E_2 has two edges repeated twice as in Fig. 10. If a and b are endpoints of a repeated edge as in Fig. 10(ii), it is equivalent to Case 4 in Fig. 9(ii). Otherwise we get the case of Fig. 10(i) equivalent to Case 3. In both cases, if $H+ab$ is non-planar, then all the other augmented side components are planar. This completes the proof.

4.2 Corollary. *If G is toroidal, then there can be at most one non-planar side component of TK_5.*

4.3 Proposition. *If all the side components of TK_5 in G are planar and at most one of the augmented side components is non-planar, then G is toroidal.*

Proof. If all the augmented side components are planar, then by Lemma 2.4 we can embed all the side components as planar graphs with two corners on the outer face in any of the embeddings of TK_5 on the torus.

Suppose one of the augmented side components of TK_5, say $H + ab$, is not planar. Then it is not possible to embed the corresponding planar side component H into an open disk with both corners a and b on the boundary. However, there are two embeddings of TK_5 on the torus (E_2 and E_5 of Fig. 6) with a side appearing exactly twice on the boundary of a face. Denote such a face by F. The face F is indicated in the diagram of E_2 and E_5 of Fig. 6, and a side which appears twice is drawn in bold. A face F with a side appearing twice on its boundary defines a cylinder. We can create a cylindrical embedding of the planar graph H as follows. Embed H on the sphere, and cut a small open disk which touches a from the interior of a face having a on its boundary, and cut another open disk which touches b from the interior of a face having b on its boundary. This converts the sphere into a finite cylinder. Now H contains an ab-path, namely the side ab of TK_5. Cut the cylinder along this ab-path to convert it into an open disk with a repeated ab-path on its boundary. This cylindrical embedding of H can then be placed in the face F of the TK_5 embedding E_2 or E_5 of Fig. 6. Any planar rotation system of H provides such a cylindrical embedding of the side component H, and vice versa.

Since all the other augmented side components of TK_5 are planar, by Lemma 2.4 any side component different from H can be embedded in an open disk with both corners on the outer face. This completes the proof.

We now consider graphs G with non-planar side components of TK_5. Before we give an equivalent of Theorem 3.1 for the torus, we show two families of graphs which can be considered as combinations of two K_5-subdivisions having at most one side in common. One family is presented in Fig. 11 and another in Fig. 12.

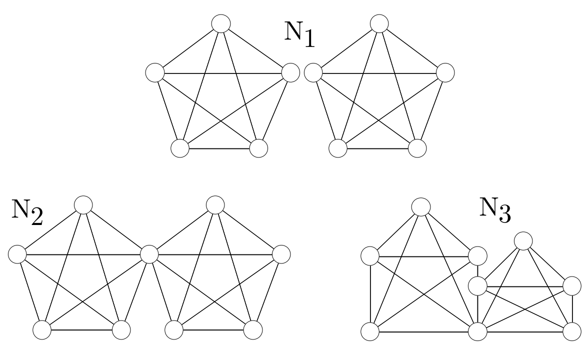

Fig. 11, Non-toroidal graphs N_1, N_2, N_3.

4.4 Lemma. *None of graphs N_1, N_2 or N_3 of Fig. 11 can be embedded on the torus.*

Proof. Consider all embeddings of K_5 on the torus (see Fig. 6). Clearly, it is not possible to embed a non-planar component K_5 into an open disk. Therefore we can not complete any of the embeddings of Fig. 6 to graph N_1. Now to extend one of the embeddings of Fig. 6 to N_2, it is necessary to embed K_4 into a face equivalent to an open disk and then add edges between all vertices of K_4 and one corner on the boundary of the face. Since K_4 is not outer-planar, it is not possible to do so without edge crossings. Therefore N_2 is non-toroidal. Finally, to extend one of the embeddings of Fig. 6 to N_3, we need to embed K_4 into a face equivalent to an open disk and then add edges between all four vertices of K_4 and two corners on the boundary of the face. Since K_4 is not outer-planar, it can not be done without edge crossings. Therefore N_1, N_2 and N_3 are not toroidal.

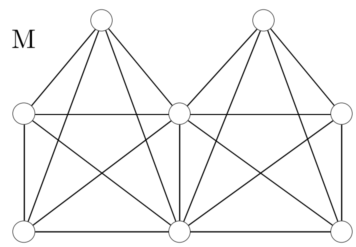

Fig. 12, Toroidal graph M.

It can be seen that we can complete some of the embeddings of K_5 on the torus to an embedding of graph M of Fig. 12. We must add a K_3 into one of the faces of an embedding of K_5, and join each vertex of K_3 to 2 corners of K_5. This can be done in several ways (using E_2, E_4 or E_5 of Fig. 6) . Notice that any embedding of M on the torus has similar properties to the embeddings of K_5 on the projective plane. We state them in the following lemma.

4.5 Lemma. *In any embedding of the graph M on the torus, every vertex of M appears at most once on the boundary of any face of the embedding and every edge of M is on the boundary of exactly two faces.*

We omit a proof of Lemma 4.5 because of the number of technical details. The proof is done by adding a triangle into a face of the torus embeddings of K_5 (E_2, E_4 and E_5 of Fig. 6) and all edges between the triangle and two corners of K_5 in all possible ways.

The graph M can be viewed as two K_5's with one edge identified. Let TM be a subdivision of M. TM contains two K_5-subdivisions, TK_5' and

TK_5'', with one side in common. A *corner* of TM is a corner of any of the two TK_5's. A *side* of TM is a side of any of the TK_5's. A vertex of TM is called *inner* if it is an inner vertex in any of the TK_5's.

Now suppose graph G has TM as a subgraph and there is no short cut or 3-corner vertex of any of two corresponding TK_5's of TM in G. Then TK_5' is contained in a side component of TK_5'' in G and vice versa. As in Proposition 2.2, let K be the set of corners of TM. We define a *side component* of TM as a subgraph in G induced by a pair of corners a and b of TK_5' or TK_5'' in TM and all connected components of $G \backslash K$ adjacent to a and b. Clearly, any two side components of TM can intersect just in the common corner of TM if one exists. An *augmented side component* of TM is defined as before. Clearly, Lemma 2.4 holds as well for the side components of TM.

The next proposition provides an alternative proof of Corollary 4.2.

4.6 Proposition. *If there is more than one non-planar side component of TK_5 in G, then G is not toroidal.*

Proof. Two side components each containing a subdivision of K_5 or $K_{3,3}$ can intersect in at most one vertex. Therefore G contains as a minor one of N_1, N_2 of Fig. 11 or one of the graphs of Fig. 13. This covers all possible combinations of K_5 and $K_{3,3}$.

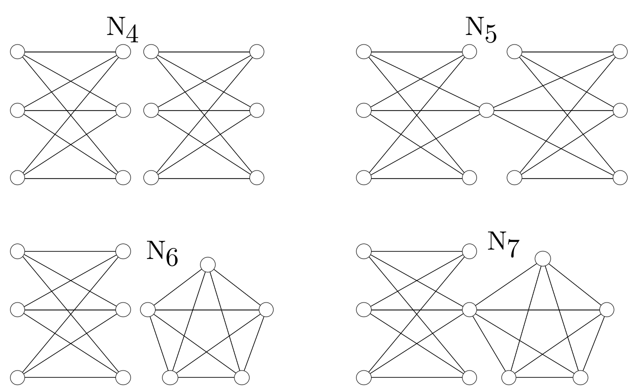

Fig. 13, Non-toroidal graphs.

Similar reasoning to Lemma 4.4 shows that graphs N_4, N_5, N_6 and N_7 of Fig. 13 are not toroidal. It is not possible to embed $K_{3,3}$ into an open disk. This rules out N_4 and N_6. Consider N_5 and N_7 as $K_{3,3}$ or K_5, respectively, with one vertex adjacent to 3 independent vertices of $K_{2,3}$. $K_{2,3}$ is not outer-planar, yet must be embedded in an open disk. There is always one vertex of a set of 3 independent vertices of $K_{2,3}$ that cannot be joined to a vertex on the boundary of the disk. Consequently, it is impossible to complete an embedding of $K_{3,3}$ to N_5 or K_5 to N_7. Thus G has a non-toroidal minor. This proves the proposition.

Therefore it remains to distinguish between toroidal and non-toroidal graphs in the case of a single non-planar side component. Let H be the unique non-planar side component of TK_5 in G. Denote the corresponding side of TK_5 by h and its corners by a and b. Suppose that H contains a K_5-subdivision TK_5' and that there is no short cut or 3-corner vertex of TK_5' in G.

4.7 Theorem. *Let graph G have a K_5-subdivision TK_5 with no short cut or 3-corner vertex in G. Let there be one non-planar side component H of TK_5 which contains a K_5-subdivision TK_5' with no short cut or 3-corner vertex in G. Then G is toroidal iff TK_5 and TK_5' have two common corners, and $TK_5 \cup TK_5'$ contains an M-subdivision TM all of whose augmented side components in G are planar.*

Proof. Sufficient conditions. In any embedding of TM on the torus, for each side of TM construct a planar embedding of its side component with both corners on the outer face. By Lemma 2.4 there exists such an embedding of a side component if and only if the augmented side component is a planar graph. Clearly, we can embed every side component independently to obtain an embedding of G.

Necessary conditions. We consider all possible cases of intersection of TK_5 and TK_5' in G. If $TK_5 \cap TK_5' = \emptyset$, then G has minor N_1 of Fig. 11 and G is not toroidal.

If $TK_5 \cap TK_5' \neq \emptyset$, let h be the side of TK_5 which is contained in a non-planar side component H, and let a and b be the corners of H. Denote by x the vertex of $h \cap TK_5'$ closest to a on side h and by y the vertex of $h \cap TK_5'$ closest to b on side h. If $x = y$, then G has minor N_2 of Fig. 11, obtained by possibly contracting the edges of a path, and so G is not toroidal. So, $x \neq y$.

Without loss of generality, suppose $x \neq a$. The following cases are possible.

1) *x is an inner vertex on a side of TK_5'.*

If y is on the same side of TK_5' as x, then G contains minor N_3 of Fig. 11 and G is not toroidal. Otherwise y is on a different side of TK_5', and TK_5 contains a short cut of TK_5' in G with endpoints x and y – a contradiction since G has no short cut of TK_5 or TK_5'.

2) *x is a corner of TK_5'.*

If y is on the same side of TK_5' as x, then G contains a minor N_3 of Fig. 11 and G is not toroidal. Otherwise y is on a different side of TK_5'.

Then y is an inner vertex of TK_5' and TK_5 contains a short cut of TK_5' in G with endpoints x and y – a contradiction.

Hence $x = a$ and $y = b$. Now suppose x or y is an inner vertex of TK_5'. If x and y are on the same side of TK_5', we have a minor N_3 of Fig. 11 in G and G is not toroidal. If x and y are on different sides of TK_5', then TK_5 contains a short cut of TK_5' in G – a contradiction. Thus x and y are both corners of TK_5'.

Without loss of generality we can substitute the side h of TK_5 by the side between x and y in TK_5'. Clearly, the substitution does not create any short cut or 3-corner vertex of TK_5 in G and it does not affect the side components of TK_5 in G. On the other hand, TK_5 and TK_5' now have a common side and give us an M-subdivision TM in G. Clearly, Lemma 2.4 holds for the side components of TM in G too. By using Lemma 4.5, the same reasoning as in Theorem 3.1 shows that an embedding of TM on the torus can be extended to G iff all the side components of TM in G are planar with both corners on the outer face. A non-planar augmented side component of TM in G can not be added into any of the embeddings of TM on the torus. This completes the proof.

Propositions 4.1 and 4.3 as well as Theorem 4.7 give us a linear time practical algorithm for graphs with a K_5-subdivision.

Torus Embedding Algorithm for Graphs with a K_5-Subdivision.

Input: *a graph G*

Output: *Either a toroidal rotation system of G, or a $K_{3,3}$-subdivision in G, or G is not toroidal*

1. Use a planarity checking algorithm (eg. [6]) to determine if G is planar. If G is planar then return its planar rotation system. If G is not planar and the planarity check returns a $K_{3,3}$-subdivision in G then return the $K_{3,3}$-subdivision in G.
2. G is not planar and the planarity check returned a K_5-subdivision TK_5 in G. Do a depth-first or breadth-first search to find either a short cut or a 3-corner vertex of TK_5 in G. If a short cut or 3-corner vertex is found, then return a $K_{3,3}$-subdivision in G. If there is no short cut or 3-corner vertex, the depth-first or breadth-first search returns the side components of TK_5.
3. If there are two non-planar augmented side components of TK_5 in G, then return G is not toroidal. If there is at most one non-planar augmented side component of TK_5 and the corresponding side components of TK_5 in G is planar, then return a toroidal rotation system of G.

4. There is exactly one non-planar side component of TK_5 in G. If the planarity check for the side component returned a $K_{3,3}$-subdivision, then return the $K_{3,3}$-subdivision in G. If the planarity check for the side component returned a K_5-subdivision TK_5', then do a depth-first or breadth-first search to check if there is a short cut or a 3-corner vertex of TK_5' in G. If a short cut or a 3-corner vertex of TK_5' is found, then return a $K_{3,3}$-subdivision in G.

5. Check if TK_5 and TK_5' have two common corners. If they do not have two common corners, then return G is not toroidal. If they do have two common corners, then construct an M-subdivision TM in G. Find the side components of TM using a depth-first or breadth-first search.

6. For each augmented side component of TM in G, check if it is planar. If all the augmented side components are planar, then return a toroidal rotation system of G. If there is a non-planar augmented side component of TM, then return G is not toroidal.

Every step in this algorithm has linear time complexity, so the entire algorithm is linear. It can be easily implemented using a planarity checking algorithm and breadth-first or depth-first searches to find the side components.

5. Conclusion

The algorithms presented here simplify algorithms in [7], [10] and [12]. Also, they are easy to implement. By using the first algorithm, we exclude 27 initial labelled embeddings of K_5 and consider just the remaining 6 labelled embeddings of $K_{3,3}$ on the projective plane in [10] and [12]. By using the second algorithm, we exclude 6 unlabelled embeddings of K_5 and need to consider just 2 unlabelled embeddings of $K_{3,3}$ on the torus for [7].

We do not know of any implementation of the algorithms of [7] and [10], nor of their generalization in [11]. The most efficient implemented algorithms we know of are presented in [12] and [13]. We hope to develop our ideas and techniques to devise practical and more efficient general algorithms. Also, the approach of excluding K_5-subdivisions can likely be generalized for graph embedding algorithms in oriented and non-oriented surfaces of higher genus.

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