On Butterfly Factorizations

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Abstract

We define a butterfly factorization for the complete graph K_{2n} and give a construction for all values of n

1 Introduction

Recently, a problem in design theory required us to use a factorization of K_8 of the following type.

37 46 58	**12**	35 47 68
18 26 57	**34**	15 28 67
17 24 38	**56**	13 27 48
16 23 45	**78**	14 25 36

The central pair (in bold) of each row forms the body of a "butterfly", with a left and right "wing". There are 8 one-factors, each consisting of a body along with either a left wing or a right wing. All pairs occur, since there are n bodies and each body, along with its two wings, uses $2n-1$ pairs.

Note that in the traditional one-factorization there are $2n-1$ one-factors, each containing n pairs; in the butterfly factorization, there are n butterflies, each containing $2n-1$ pairs.

This factorization of K_8 was easily found experimentally, and an analogous factorization for K_6 is even easier to produce, namely,

36 45	**12**	35 46
16 25	**34**	15 26
14 23	**56**	13 24

Obviously, this butterfly factorization of K_6 is unique.

Since the number of ordinary one-factorizations of K_{2n} increases very rapidly with n, it seemed likely that a butterfly factorization could be obtained for arbitrary n.

2 Construction for the case of K_{2n}, n odd

Write down the points of K_n twice; in the first line, these are written as

$$1\ 2\ 3\ \ldots\ n,$$

and in the second line they are written as

$$n+1, n+2, n+3, \ldots, n+n.$$

The pairs $(1, n+1), (2, n+2), (3, n+3), \ldots, (n, n+n)$ will form the n butterfly bodies.

Now write down a near one-factorization of K_n, using the symbols $1, 2, \ldots, n$. This will produce n sets, each set comprising $(n-1)/2$ pairs and a singleton, the singletons all being different.

Let this near one-factorization be

$$(u_1, u_2), (u_3, u_4), \ldots, (u_{t-1}, u_t), i,$$

We then construct the butterfly with body $(i, n+1)$ by placing (u_j, u_{j+1}) and $(u_j+n, u_{j+1}+n)$ in one wing and $(u_j, u_{j+1}+n)$ and (u_{j+1}, u_j+n) in the other wing.

Two examples will make the procedure clear.

Example 1. K_{10}.

The K_5 is on 1,2,3,4,5, and the near one-factorization is:

$$(2,3),(4,5),1 \qquad (1,4),(3,5),2 \qquad (1,5),(2,4),3$$
$$(1,3),(2,5),4 \qquad (1,2),(3,4),5$$

We then apply the algorithm described and obtain

$$\begin{array}{rcl}
(2,3)\ (7,8)\ (4,5)\ (9,10) & (\mathbf{1},\mathbf{6}) & (2,8)\ (3,7)\ (4,10)\ (5,9) \\
(1.4)\ (6,9)\ (3,5)\ (8,10) & (\mathbf{2},\mathbf{7}) & (1,9)\ (4,6)\ (3,10)\ (5,8) \\
(1,5)\ (2,4)\ (6,10)\ (7,9) & (\mathbf{3},\mathbf{8}) & (1,10)\ (5,6)\ (2,9)\ (4,7) \\
(1,3)\ (2,5)\ (6,8)\ (7,10) & (\mathbf{4},\mathbf{9}) & (1,8)\ (3,6)\ (2,10)\ (5,7) \\
(1,2)\ (3,4)\ (6,7)\ (8,9) & (\mathbf{5},\mathbf{10}) & (1,7)\ (2,6)\ (3,9)\ (4,8)
\end{array}$$

Example 2. K_{14}.

The first K_7 is on symbols 1,2,3,4,5,6,7, and the near one-factorization is

$$\begin{array}{ll}
(2,7),(3,6),(4,5),1 & (1,3),(4,7),(5,6),2 \\
(2,4),(1,5),(6,7),3 & (3,5),(2,6),(1,7),4 \\
(4,6),(3,7),(1,2),5 & (5,7),(1,4),(2,3),6 \\
(1,6),(2,5),(3,4),7 &
\end{array}$$

Apply the algorithm to give the following butterfly factorization.

$$\begin{array}{rcl}
(2,7)\ (9,14)\ (3,6)\ (10,13)\ (4,5)\ (11,12) & (\mathbf{1},\mathbf{8}) & (2,14)\ (7,9)\ (3,13)\ (6,10)\ (4,12)\ (5,11) \\
(1,3)\ (8,10)\ (4,7)\ (11,14)\ (5,6)\ (12,13) & (\mathbf{2},\mathbf{9}) & (1,10)\ (3,8)\ (4,14)\ (7,11)\ (5,13)\ (6,12) \\
(2,4)\ (9,11)\ (1,5)\ (8,12)\ (6,7)\ (13,14) & (\mathbf{3},\mathbf{10}) & (2,11)\ (4,9)\ (1,12)\ (5,8)\ (6,14)\ (7,13) \\
(3,5)\ (10,12)\ (2,6)\ (9,13)\ (1,7)\ (8,14) & (\mathbf{4},\mathbf{11}) & (3,12)\ (5,10)\ (2,13)\ (6,9)\ (1,14)\ (7,8) \\
(4,6)\ (11,13)\ (3,7)\ (10,14)\ (1,2)\ (8,9) & (\mathbf{5},\mathbf{12}) & (4,13)\ (6,11)\ (3,14)\ (7,10)\ (1,9)\ (2,8) \\
(5,7)\ (12,14)\ (1,4)\ (8,11)\ (2,3)\ (9,10) & (\mathbf{6},\mathbf{13}) & (5,14)\ (7,12)\ (1,11)\ (4,2)\ (2,10)\ (3,9) \\
(1,6)\ (8,13)\ (2,5)\ (9,12)\ (3,4)\ (10,11) & (\mathbf{7},\mathbf{14}) & (1,13)\ (6,8)\ (2,12)\ (5,9)\ (3,11)\ (4,10)
\end{array}$$

3 Construction for K_{2n}, $n \equiv 0$, modulo 4.

We start by considering n even. It will be convenient to use symbols $0, 1, \ldots, 2n-1$ when n is even and to use the set of bodies as the pairs $(i, n+i)$ as i ranges from 0 to n. We first suppose that we have a wing to attach to $(0, n)$ and that this wing contains $(n-1)$ pairs (i, j) where $i - j$ takes on all the values $1, 2, \ldots, n-1$, modulo $2n$.

If such a wing is possible, we have $2n-2$ symbols available for i and j, with n of these being odd $(1, 3, 5, \ldots, 2n-1)$ and $n-2$

being even $(2, 4, \ldots, n-2, n+2, \ldots, 2n-2)$. We wish to pair these symbols so as to have differences $1, 2, \ldots, n-1$. Of these differences, $n/2$ are odd and $n/2-1$ are even. To get the odd differences, we must use $n/2$ odd numbers and $n/2$ even numbers. This leaves $n/2$ odd numbers and $n/2-2$ even numbers to form even differences. But even differences require pairs in which both elements of the pair have the same parity. Hence $n \equiv 0$, modulo 4, and the $n/2$ odd numbers produce $n/4$ even differences, whereas the $n/2-2$ even numbers produce $n/4-1$ even differences. Thus the total number of even differences is $n/2-1$, as required. We state our results as

Lemma 1. *If* $(0, n)$ *has a wing comprising pairs* (i, j) *where* $i - j$ *assumes all values* $1, 2, \ldots, n-1$, *then* $n \equiv 0$ modulo 4.

If the condition of Lemma 1 is satisfied, then the other wing attached to the body $(0, n)$ is simply the set of pairs $(i+n, j+n)$, and the total butterfly factorization can be obtained from this one butterfly by adding the numbers $1, 2, 3, \ldots, n-1$, successively, to the initial butterfly.

We first give a simple example for $2n = 8$, and then provide a general solution for $n = 4t$.

Example 3. $2n = 8$.

It is easy to write down one wing to accompany the body $(0, 4)$ as $(1, 2), (5, 7), (3, 6)$. Then the initial butterfly is (remember that we are working mod 8):

$$(5,6)\ (1,3)\ (7,2)\quad (\mathbf{0},\mathbf{4})\quad (1,2)\ (5,7)\ (3,6)$$

The other butterflies are obtained by successive additions of unity to give

$$\begin{array}{lll}
(6,7)\ (2,4)\ (0,3) & (\mathbf{1},\mathbf{5}) & (2,3)\ (6,0)\ (4,7)\\
(7,0)\ (3,5)\ (1,4) & (\mathbf{2},\mathbf{6}) & (3,4)\ (7,1)\ (5,0)\\
(0,1)\ (4,6)\ (2,5) & (\mathbf{3},\mathbf{7}) & (4,5)\ (0,2)\ (6,1)
\end{array}$$

Consider now the general case $2n = 8t$ (i.e., $n \equiv 0 \pmod 4$). We need merely create the first wing that is attached to $(0, n)$. We want all differences $1, 2, \ldots, 4t-1$, modulo $8t$.

First, take the pairs $(i, 4t-i)$ where i ranges from 1 to $2t-1$. This produces even differences $4t-2i$ that range from 2 to $4t-2$.

Then take differences $(4t+k, 8t-k+1)$ where k ranges from 2 to $t-1$. This produces odd differences $4t-2k+1$ that range from $4t-3$ to $2t+3$. Then take pairs $(6t-i, 6t+i+3)$ where i ranges from 0 to $t-2$. This produces odd differences $2i+3$ that range from 3 to $2t-1$.

We now have all differences except 1, $4t-1$ and $2t+1$. We obtain difference 1 from the pair $(5t, 5t+1)$. The difference $4t-1$ arises from the pair $(2t, 6t+1)$, since $4t-1 \equiv -(4t+1)$, mod $8t$, so that it represents the same difference. The difference $2t+1$ arises from the pair $(4t+1, 6t+2)$.

Example 4. $2n = 64$ $(t = 8)$.

The initial body is $(0, 32)$. The successive sets of pairs in the algorithm just described are:

$$\begin{array}{rl}
(1,31),(2,30),(3,29),\ldots,(15,17) & [30,28,\ldots,2] \\
(34,63),(35,62),\ldots,(39,58) & [29,27,\ldots,19] \\
(48,51),(47,52),\ldots,(42,57) & [3,5,\ldots,15] \\
(40,41),(16,49),(33,50) & [1,31,17]
\end{array}$$

The numbers in square brackets are the differences.

Once one has this wing, the other wing is obtained by adding 32 (mod 64), and the other butterflies are obtained, as in Example 3, by successive additions of unity. It can be verified that in general, the $4t-1$ pairs described above contain each of the numbers $1, 2, \ldots, 4t-1, 4t+1, \ldots, 8t-1$, exactly once. We have seen that they contain all the required differences. This gives:

Lemma 2. *When $n \equiv 0 \pmod 4$, K_{2n} has a butterfly factorization.*

4 Construction for K_{2n}, $n \equiv 2 \pmod 4$

We now handle the remaining possibilities: $n \in \{6, 10, 14, 18, \ldots\}$. Thus $2n$ is of the form $4t$, where t is an odd integer greater than one. By Lemma 1, it is not possible to construct a single wing containing all differences, and then successively add one to obtain the remaining butterflies. We will show that it is possible to construct two butterflies which together contain all differences exactly twice. We will then successively add one to the initial butterflies to obtain the factorization.

We start with an example.

Example 5. K_{12} $(4n = 12, n = 3)$.

Take the bodies as (0,6), (1,7), (2,8), (3,9), (4,10), (5,11). Create a wing for (0,6) as follows.

$$(\mathbf{0}, \mathbf{6}),\ (2,3),\ (8,9),\ (1,4),\ (5,10),\ (7,11).$$

Note that we have differences 1^2,3,4,5.

Add 3 to each number to create a partial butterfly

$$(\mathbf{3}, \mathbf{9}),\ (5,6),\ (0,11),\ (4,7),\ (1,8),\ (2,10)$$

Add 3 again to give

$$(\mathbf{0}, \mathbf{6}),\ (8,9),\ (2,3),\ (7,10),\ (4,11),\ (1,5)$$

and then interchange symbols 3 and 9 with 0 and 6, respectively, to create

$$(\mathbf{3}, \mathbf{9}),\ (8,6),\ (2,0),\ (7,10),\ (4,11),\ (1,5)$$

This creates the missing difference 2 (twice). We now have each of the differences $1, 2, 3, 4, 5$ exactly twice in the wings occurring with the body $(3, 9)$. It would have been possible to have the first wing missing the difference 4 rather than 2 (if so, the symbols 6 and 0 would be used as replacements so as to create difference 4). We now have a complete butterfly for (3,9).

$$(6,8)\ (0,2)\ (7,10)\ (4,11)\ (1,5)\quad (\mathbf{3}, \mathbf{9})\quad (5,6)\ (0,11)\ (4,7)\ (1,8)\ (2,10)$$

Add 3 again to the left wing just created in order to obtain the left wing for the first wing created for (0,6); this gives a second butterfly

$$(9,11)\ (3,5)\ (1,10)\ (2,7)\ (4,8)\quad (\mathbf{0}, \mathbf{6})\quad (2,3)\ (8,9)\ (1,4)\ (5,10)\ (7,11).$$

It also has each of the differences $1, 2, 3, 4, 5$ exactly twice in its wings. These two basic butterflies produce the other four butterflies by adding 1 and 2 to each (1 to $n-1$, in general). Our complete factorization is thus:

$(9,11)$ $(3,5)$ $(1,10)$ $(2,7)$ $(4,8)$	$(\mathbf{0},\mathbf{6})$	$(2,3)$ $(8,9)$ $(1,4)$ $(5,10)$ $(7,11)$
$(0,10)$ $(4,6)$ $(2,11)$ $(3,8)$ $(5,9)$	$(\mathbf{1},\mathbf{7})$	$(3,4)$ $(9,10)$ $(2,5)$ $(6,11)$ $(0,8)$
$(1,11)$ $(5,7)$ $(0,3)$ $(4,9)$ $(6,10)$	$(\mathbf{2},\mathbf{8})$	$(4,5)$ $(10,11)$ $(3,6)$ $(0,7)$ $(1,9)$
$(6,8)$ $(0,2)$ $(7,10)$ $(4,11)$ $(1,5)$	$(\mathbf{3},\mathbf{9})$	$(5,6)$ $(0,11)$ $(4,7)$ $(1,8)$ $(2,10)$
$(7,9)$ $(1,3)$ $(8,11)$ $(0,5)$ $(2,6)$	$(\mathbf{4},\mathbf{10})$	$(6,7)$ $(0,1)$ $(5,8)$ $(2,9)$ $(3,11)$
$(8,10)$ $(2,4)$ $(0,9)$ $(1,6)$ $(3,7)$	$(\mathbf{5},\mathbf{11})$	$(7,8)$ $(1,2)$ $(6,9)$ $(3,10)$ $(0,4)$

This example shows that we need merely create the first body and wing, starting from $(0,2n)$ with pairs $(n-1,n)$ and $(3n-1,3n)$, and using the other symbols to create pairs with all differences 2 to $2n-1$, omitting difference $n-1$ (or difference $n+1$). The algorithm of Example 5 can then proceed.

Our initial body is $(\mathbf{0},\mathbf{2n})$.

It will be convenient to write $n-1=2a$. Then we create our general pairs as follows:

(a) $(n,n-1)$ and $(3n-1,3n)$ give difference $[1]$ twice, where we use square brackets to denote differences.

(b) Pairs $(i,2n-i)$ produce differences $[2n-2i]$ where i ranges from 1 to a; this produces all even differences from $[2n-2]$ down to $[2n-2a]=[n+1]$.

(c) Pairs $(2n+a+i,2n+a+3i)$ produce all even differences $[2i]$ from $[2]$ up to $[n-5]$ as i ranges from 1 to $a-2$.

(d) Pairs $(2a-i,n+i)$ produces odd differences $[n-2a+2i]=[2i+1]$ ranging from $[3]$ to $[n-2]$ as i ranges from 1 to $a-1$.

(e) Pairs $(2n+a-i,3n+a+i)$ produces odd differences $[n+2i]$ ranging from $[n]$ to $[2n-3]$ as i ranges from 0 to $a-1$.

(f) Difference $2n-1$ is produced from the pair $(n+a,3n+a-1)$. Difference $n-3$ is produced from the pair $(3n+2,4n-1)$.

We now have the differences $1^2,2,\ldots,2n-1$, except for the even difference $n-2$. The pairs occurring with the body $(\mathbf{0},\mathbf{2n})$ contain all numbers $1, 2, \ldots, 2n-1$, except for n. We give an example for the creation of the first wing for the case $n = 9$ (K_{36}) ($a = 4$).

Example 6. Initial body $(\mathbf{0}, \mathbf{18})$.

(a) Pairs (8,9) and (26,27).

(b) Pairs (1,17), (2,16), (3,15), (4,14).

(c) Pairs (23,25), (24,28).

(d) Pairs (7,10), (6,11), (5,12).

(e) Pairs (22,31), (21,32), (20,33), (19,34).

(f) Pairs (13,30), (29,35).

We now add n to each number to obtain an initial wing for the body $(\mathbf{n}, \mathbf{3n})$ having the same differences. We add n again to obtain another set of pairs for the body $(\mathbf{2n}, \mathbf{0})$. Interchange the symbols n and $3n$ with 0 and $2n$, respectively. This gives the second wing for the body $(\mathbf{n}, \mathbf{3n})$. The pairs containing $0, n, 2n$, or $3n$ are $(3n, 3n-1)$ and $(n-1, n)$, which become $(2n, 3n-1)$ and $(n-1, 0)$ after interchanging symbols. This changes the differences $[1^2]$ to $[2^2]$. Thus the two wings for $(\mathbf{n}, \mathbf{3n})$ together contain each difference exactly twice. We now add n again to obtain the second wing for $(\mathbf{0}, \mathbf{2n})$. Its two wings also contain each difference exactly twice. By successively adding $1, 2, \ldots, n-1$ we obtain a complete butterfly factorization. This gives:

Lemma 3. *When $n \equiv 2 \pmod 4$, K_{2n} has a butterfly factorization.*

5 An Alternative for K_{4t}

The algorithm in Section 4 is not restricted to the particular beginning wing used for the general proof. Examples of a different initial wing are given in this section.

K_{20}: $(\mathbf{0}, \mathbf{10})$ $(4, 5)$ $(14, 15)$ $(11, 13)$ $(3, 6)$ $(1, 7)$ $(8, 16)$ $(9, 18)$ $(2, 17)$

K_{28}: $(\mathbf{0},\mathbf{14})$ $(6,7)$ $(20,21)$ $(11,22)$ $(12,24)$ $(13,26)$ $(16,18)$ $(15,19)$
$(2,23)$ $(3,25)$ $(1,10)$ $(4,9)$ $(5,8)$ $(17,27)$

K_{36}: $(\mathbf{0},\mathbf{18})$ $(8,9)$ $(26,27)$ $(14,28)$ $(15,30)$ $(16,32)$ $(17,34)$ $(1,13)$
$(22,35)$ $(21,23)$ $(20,24)$ $(19,25)$ $(4,29)$ $(2,31)$ $(5,33)$ $(3,12)$
$(6,11)$ $(7,10)$

6 Concluding Remarks

We pointed out earlier that the butterfly factorization of K_6 is unique. The number of non-isomorphic butterfly factorizations for K_8 is very limited, but the number increases rapidly for K_{2n} as n increases. For example, the number of non-isomorphic butterfly factorizations of K_{2n} (n odd) is certainly at least as great as the number of non-isomorphic one-factorizations of K_{n+1}, in virtue of the construction used in Section 2.