SOME CONSTRUCTIONS FOR K-RECONSTRUCTION

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1. *Introduction.*

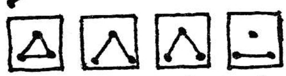
The purpose of this article is to exhibit some constructions associated with the idea of k-reconstruction, as described in [2], and thereby to deduce some bounds for the function $\nu_R(k)$ defined in [2]. We also include a list of all known non-3-reconstructible unordered triples of graphs.

We begin by stating the appropriate definitions. Undefined symbols or terms may be found in [1] and/or [2].

DEFINITION: The *deck* , D[G], of a graph G is the multiset consisting of the collection of vertex-deleted subgraphs of G. Each vertex-deleted subgraph is called a *card* of G.

Example:

G = 

D[G] = 

DEFINITION: Let $G_1, G_2, \ldots, G_k$ be k graphs such that $\nu(G_1) = \nu(G_2) = \ldots = \nu(G_k)$. The *shuffled k-deck*, $D[G_1, G_2, \ldots, G_k]$, of the unordered k-tuple of graphs $[G_1, G_2, \ldots, G_k]$ is the multiset formed by shuffling together the individual decks $D[G_1], D[G_2], \ldots, D[G_k]$.

DEFINITION: An unordered k-tuple of graphs, $[G_1, G_2, \ldots, G_k]$ is k-reconstructible if there do not exist graphs $H_1, H_2, \ldots, H_k$ such that $D[G_1, G_2, \ldots, G_k] = D[H_1, H_2, \ldots, H_k]$, but $[G_1, G_2, \ldots, G_k] \neq [H_1, H_2, \ldots, H_k]$.

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If Γ is a class of graphs (e.g., trees, disconnected graphs, etc.) such that all unordered k-tuples of graphs from Γ are k-reconstructible, we say that Γ is k-reconstructible.

The following two theorems were proved in [2].

THEOREM 1. *Let* G_1 *and* G_2 *be disconnected graphs such that* $\nu(G_1) = \nu(G_2) \geq 5$ *and* $\omega(G_1) + \omega(G_2) \geq 6$. *Then* $[G_1, G_2]$ *is 2-reconstructible.*

THEOREM 2. *The class of disconnected graphs with at least five vertices is 2-reconstructible if and only if the reconstruction conjecture is true.*

These two theorems suggest that perhaps all graphs with at least five vertices are 2-reconstructible if the reconstruction conjecture is true. Indeed, it was found (see [2]) that although there are several non-2-reconstructible pairs of graphs on two, three and four vertices, there are none on five or six vertices. It proved too difficult to examine the 1044 graphs on seven vertices.

It was suggested in [2] that there exists a function $\nu_R(k)$ such that all graphs with at least $\nu_R(k)$ vertices are k-reconstructible. In particular, $\nu_R(1) = 3$ corresponds to the reconstruction conjecture, and $\nu_R(2) = 5$ is strongly suggested by [2]. We show below that $\nu_R(k) \geq 3 + 2\,[\log k]$, and we suspect that in fact, $\nu_R(k) = 3 + 2\lceil \log k \rceil$, where the upper brackets, $\lceil x \rceil$, represent the least integer greater than or equal to x. Here, and in the sequel, all logarithms are taken to base two.

k	$3+2\lfloor\log k\rfloor$	$3+2\lceil\log k\rceil$
1	3	3
2	5	5
3	5	7
4	7	7
5	7	9
6	7	9
7	7	9
8	9	9

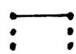
Table 1.

2. *Non-K-Reconstructible Families.*

Let $H_{m,n}$ denote the graph on $2m$ vertices formed by taking the disjoint union of n edges and $2(m-n)$ isolated points.

Example:

$H_{3,3}$ = 

$H_{3,1}$ = 

THEOREM 3. *Let $m \geq 1$. Let $k = 2^{m-1}$. Let G represent the unordered k-tuple of graphs containing*

$\binom{m}{0}$ *copies of* $H_{m,m}$,

$\binom{m}{2}$ *copies of* $H_{m,m-2}$,

$\binom{m}{4}$ *copies of* $H_{m,m-4}$,

$\ldots$,

$\binom{m}{2[m/2]}$ *copies of* $H_{m,m-2[m/2]}$.

Let H *represent the unordered* k*-tuple of graphs containing:*

$\binom{m}{1}$ *copies of* $H_{m,m-1}$,

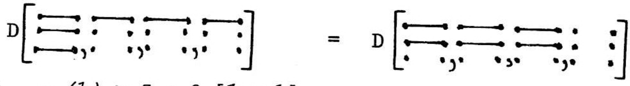
$\binom{m}{3}$ *copies of* $H_{m,m-3}$,

$\ldots$,

$\binom{m}{2[(m+1)/2]-1}$ *copies of* $H_{m,m-2[(m+1)/2]+1}$.

Then $D[G] = D[H]$, *but* $G \neq H$.

Proof: The proof is best illustrated by an example. Let $m = 3$. Then the theorem states that



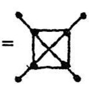
COROLLARY 4. $\nu_R(k) \geq 3 + 2\,[\log k]$.

Proof: The construction of Theorem 3 provides a non-2^{m-1}-reconstructible set of graphs on $2m$ vertices. Thus $\nu_R(2^{m-1}) \geq 2m+1$. If $k = 2^{m-1}$, this implies that $\nu_R(k) \geq 3 + 2\,[\log k]$, where we have used the fact that $\nu_R(k+1) \geq \nu_R(k)$ (see [2]).

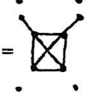
Now let $X_{m,n}$ denote the graph on $2m$ vertices formed by choosing a complete m-graph, K_m, and appending a vertex of degree one to n of its vertices, and then adding $m-n$ isolated points.

Example:

$X_{4,4}$ =



$X_{4,2}$ =



THEOREM 5. *Let* $m \geq 1$. *Let* $k = 2^{m-1}$. *Let* G *represent the unordered* k*-tuple of graphs containing:*

$\binom{m}{0}$ *copies of* $X_{m,m}$,

$\binom{m}{2}$ *copies of* $X_{m,m-2}$,

. . . ,

$\binom{m}{2[m/2]}$ *copies of* $X_{m,m-2[m/2]}$.

Let H *represent the unordered k-tuple of graphs containing:*

$\binom{m}{1}$ *copies of* $X_{m,m-1}$,

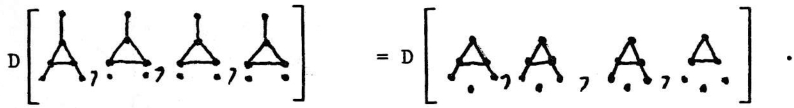
$\binom{m}{3}$ *copies of* $X_{m,m-3}$,

. . . ,

$\binom{m}{2[(m+1)/2]-1}$ *copies of* $X_{m,m-2[(m+1)/2]+1}$.

Then $D[G] = D[H]$, *but* $G \neq H$.

Proof: This construction is very similar to that of Theorem 3. If m = 3, say, the theorem states that



This is easily seen to be true.

Note that like Theorem 3, Theorem 5 also constructs a non-2^{m-1}-reconstructible set of graphs on $2m$ vertices.

Referring to the preceeding construction, note that there are $\binom{m}{n}$ graphs with n isolated points. Since

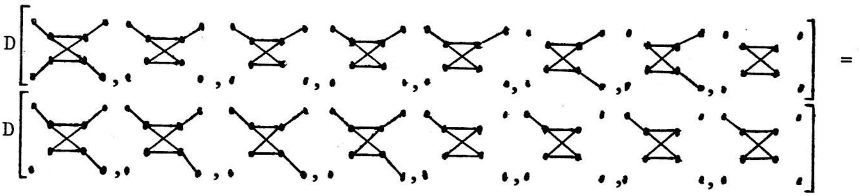
$$(m-n)\binom{m}{n} = (n+1)\binom{m}{n+1},$$

we are assured that the shuffled k-decks $D[G]$ and $D[H]$ are indeed equal. We may extend Theorem 5 by extending the above binomial identity to a multinomial identity:

$$(n_1+1)\binom{m}{n_1+1,n_2,n_3} = (n_2+1)\binom{m}{n_1,n_2+1,n_3} = (n_3+1)\binom{m}{n_1,n_2,n_3+1}.$$

This is done by replacing the complete graph K_m used in forming $X_{m,n}$ by a complete multipartite graph $K_{n_1,n_2,\ldots}$ where $n_1+n_2+\ldots = m$.

Example: Replace K_4 with $K_{2,2}$.



Naturally, we need not have $n_1 = n_2 = \ldots$. It is clear that these multipartite families all produce non-2^{m-1}-reconstructible sets of graphs on $2m$ vertices.

THEOREM 6. *Let* $D[G_1,G_2,\ldots,G_{k_1}] = D[H_1,H_2,\ldots,H_{k_1}]$ *and* $D[G_1',G_2',\ldots,G_{k_2}'] = D[H_1',H_2',\ldots,H_{k_2}']$ *be examples of non-*k_1*-reconstructible and non-*k_2*-reconstructible sets of graphs on* ν_1 *and* ν_2 *vertices,* *respectively. Then we may multiply these examples together to produce the following non-$2k_1k_2$- reconstructible set of graphs on $\nu_1 + \nu_2$ vertices. The plus sign used below indicates the operation of disjoint union of graphs.*

$$
\begin{aligned}
D[&G_1 + G_1', G_1 + G_2', \ldots, G_1 + G_{k_2}', \\
&G_2 + G_1', G_2 + G_2', \ldots, G_2 + G_{k_2}', \\
&\ldots, \\
&G_{k_1} + G_1', G_{k_1} + G_2', \ldots, G_{k_1} + G_{k_2}', \\
&H_1 + H_1', H_1 + H_2', \ldots, H_1 + H_{k_2}', \\
&\ldots, \\
&H_{k_1} + H_1', H_{k_1} + H_2', \ldots, H_{k_1} + H_{k_2}'] = \\
D[&G_1 + H_1', G_1 + H_2', \ldots, G_1 + H_{k_2}', \\
&\ldots, \\
&G_{k_1} + H_1', G_{k_1} + H_2', \ldots, G_{k_1} + H_{k_2}', \\
&H_1 + G_1', H_1 + G_2', \ldots, H_1 + G_{k_2}', \\
&\ldots, \\
&H_{k_1} + G_1', H_{k_1} + G_2', \ldots, H_{k_1} + G_{k_2}'] .
\end{aligned}
$$

Proof: Each G_i occurs once with each of $G_1', G_2', \ldots, G_{k_2}'$ on the LHS and once with each of $H_1', H_2', \ldots, H_{k_2}'$ on the RHS. Note that $D[G_1', G_2', \ldots, G_{k_2}'] = D[H_1', H_2', \ldots, H_{k_2}']$. Similarly each G_i' occurs once with each of $G_1, G_2, \ldots, G_{k_1}$ on the LHS, and once with each of $H_1, H_2, \ldots, H_{k_1}$ on the RHS. Note that $D[G_1, G_2, \ldots, G_{k_1}] = D[H_1, H_2, \ldots, H_{k_1}]$.

Similar observations apply to H_i and H_i'.

COROLLARY 7. $\nu_R(2k_1k_2) \geq \nu_R(k_1) + \nu_R(k_2) - 1.$

Proof: By multiplying a non-k_1-reconstructible set of graphs on $\nu_R(k_1) - 1$ vertices with a non-k_2-reconstructible set on $\nu_R(k_2) - 1$ vertices, we obtain a non-$2k_1k_2$-reconstructible set on $\nu_R(k_1)+\nu_R(k_2)-2$ vertices. Thus $\nu_R(2k_1k_2) \geq \nu_R(k_1) + \nu_R(k_2) - 1$.

COROLLARY 8. $\nu_R(2k^2) \geq 2\nu_R(k) - 1.$

COROLLARY 9. $\nu_R(2k) \geq \nu_R(k) + 2.$

Proof: Set $k_1 = k$ and $k_2 = 1$ in Corollary 7.

Note that by iterating Corollary 9 beginning with $\nu_R(1) = 3$, we obtain Corollary 4, namely $\nu_R(k) \geq 3 + 2\,[\log k]$. We can improve this a bit, as follows.

COROLLARY 10. $\nu_R(k) \geq \max\{7+6[(\log k/3)/\log 6],\ 3+6[\log k/\log 6],\ 5+6[(\log k/2)/\log 6]\}.$

Proof: The example

demonstrates that $\nu_R(3) \geq 7$. Substituting this into Corollary 7 yields

$$\nu_R(6k) \geq \nu_R(k) + 6.$$

If we now iterate this, beginning with $\nu_R(3) \geq 7$, we obtain

$$\nu_R(3\cdot 6^n) \geq 6n+7,$$

which holds for $n \geq 0$. We now apply Corollary 9 twice to obtain

$$\nu_R(6^{n+1}) \geq 6n+9$$

$$\nu_R(2\cdot 6^{n+1}) \geq 6n+11.$$

We now invert these three inequalities to obtain the stated result.

Note that in Theorem 3, if $m = 1$ the construction gives $D[K_2] = D[2K_1]$. We may thus consider the non-reconstructible unordered k-tuples of Theorem 3 as powers of the equality $D[K_2] = D[2K_1]$, where the powers are formed through the multiplication of Theorem 6.

3. *Computer Generated K-Tuples.*

A complete list of known non-2-reconstructible unordered pairs of graphs appears in [2]. In the following, we have used the computer to find all non-3-reconstructible unordered triples of graphs on 2,3,4,5 and 6 vertices. The following points should be kept in mind.

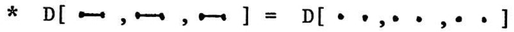
1) If $D[G_1,G_2,G_3] = D[H_1,H_2,H_3]$, then $D[\bar{G}_1,\bar{G}_2,\bar{G}_3] = D[\bar{H}_1,\bar{H}_2,\bar{H}_3]$, where the bar denotes the complement of a graph. Unless such an equality is self-complementary, we do not include the complement. Self-complementary examples are marked by an asterisk.

2) We do not include examples of the form $D[G,G_1,G_2] = D[G,H_1,H_2]$, as

these all may be constructed from the non-2-reconstructible pairs

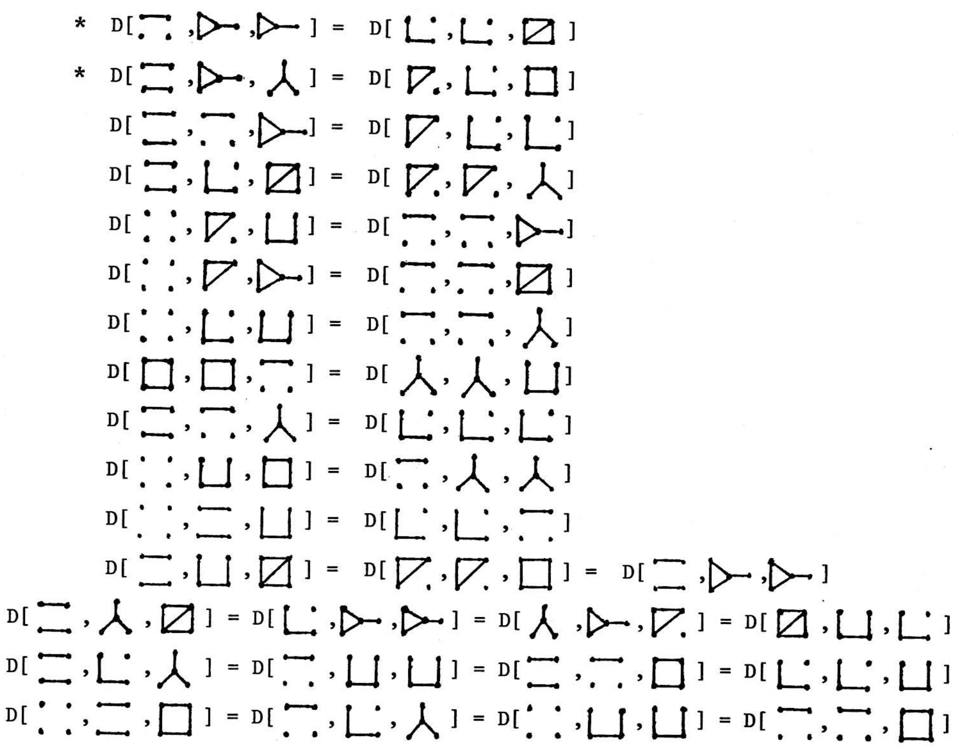
$D[G_1,G_2] = D[H_1,H_2]$.

$\nu = 2$.

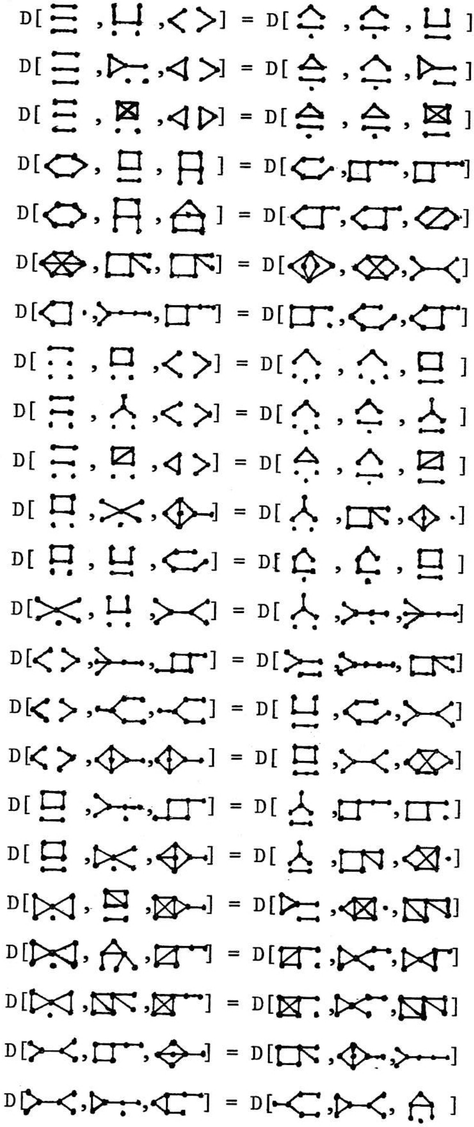


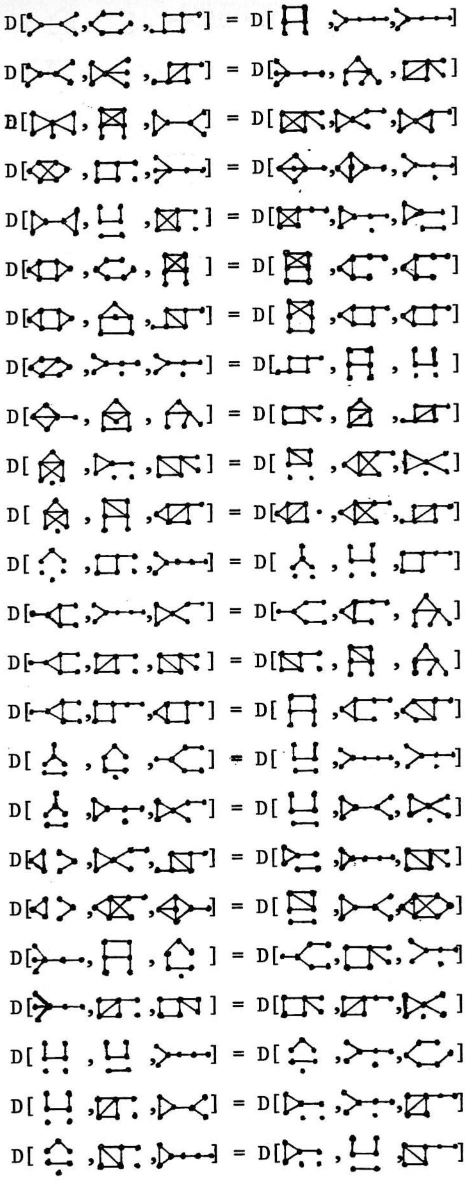
$\nu = 3$.

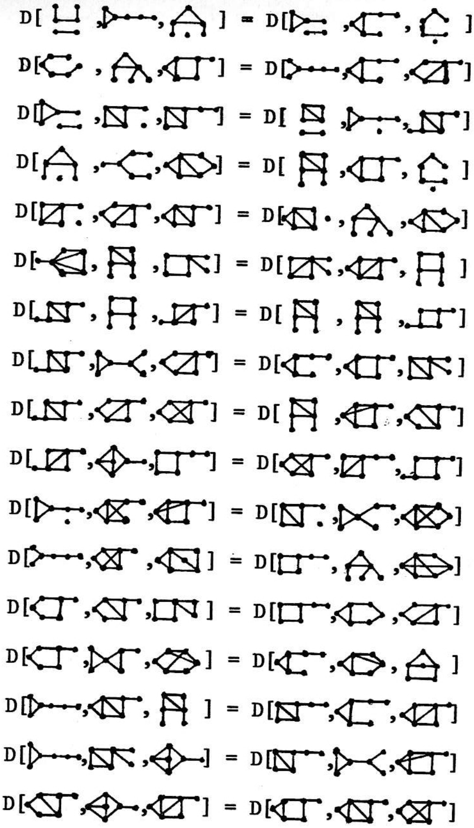
$\nu = 4$.



$\nu = 5.$







In order to construct the above examples of non-reconstructible triples, a complete list of graphs with six or fewer vertices was necessary. The tables of graphs in [3] were used for this purpose.

REFERENCES

[1] J.A. Bondy and U.S.R. Murty, *Graph Theory with Applications*, American Elsevier Publishing Co., New York, 1976.

[2] W.L. Kocay and A.H. Ball, *2-reconstruction of disconnected graphs*, Ars Combinatoria, 6 (1978), pp. 223-253.

[3] F. Harary, *Graph Theory*, Addison Wesley, Reading, Mass. (1969).

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