FINITE 2-GEODESIC TRANSITIVE GRAPHS OF VALENCY $3p$

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ABSTRACT. For a non-complete graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*geodesic* if $u \neq w$ and u, w are not adjacent. Then Γ is said to be 2-*geodesic transitive* if its automorphism group is transitive on both arcs and 2-geodesics. In this paper, we classify the family of connected 2-geodesic transitive graphs of valency $3p$ where p is an odd prime.

1. INTRODUCTION

In this paper, all graphs are finite, connected, simple and undirected. For a graph Γ, we use $V(\Gamma)$ and $\mathrm{Aut}(\Gamma)$ to denote its *vertex set* and *the automorphism group*, respectively. In a non-complete graph Γ, a vertex triple (u, v, w) with v adjacent to both u and w is called a 2-*arc* if $u \neq w$, and a 2-*geodesic* if in addition u, w are not adjacent. An *arc* is an ordered pair of adjacent vertices. The graph Γ is said to be 2-*arc transitive* or 2-*geodesic transitive* if its automorphism group $\mathrm{Aut}(\Gamma)$ is transitive on arcs, and also transitive on 2-arcs or 2-geodesics, respectively. Clearly, every 2-geodesic is a 2-arc, but some 2-arcs may not be 2-geodesics. If Γ has girth 3 (length of the shortest cycle is 3), then the 2-arcs contained in 3-cycles are not 2-geodesics. The graph in Figure 1 is the complete multipartite graph $\mathrm{K}_{4[3]}$ which is 2-geodesic transitive but not 2-arc transitive with valency 9. Thus the family of non-complete 2-arc transitive graphs is properly contained in the family of 2-geodesic transitive graphs.

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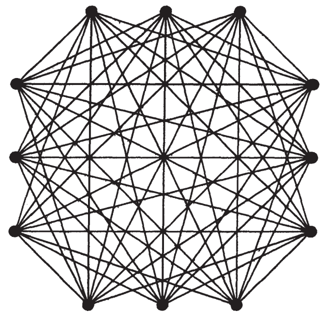

FIGURE 1. $K_{4[3]}$

The first remarkable result about 2-arc transitive graphs comes from Tutte [18, 19], and this family of graphs has been studied extensively, see [1, 8, 10, 12, 15, 21]. The local structure of the family of 2-geodesic transitive graphs was determined in [4]. In [5], Devillers, Li, Praeger and the author classified 2-geodesic transitive graphs of valency 4. Later, in [7], they completely determined the family of prime valency 2-geodesic transitive graphs. They proved that either such a graph is 2-arc transitive or the valency p satisfies $p \equiv 1 \pmod 4$, and for each such prime there is a unique graph with this property: it is a non-bipartite antipodal double cover of the complete graph K_{p+1} with automorphism group $PSL(2,p) \times \mathbb{Z}_2$ and diameter 3. In [9], the author classified the family of 2-geodesic transitive graphs of valency twice a prime, and completely determined such graphs which are locally primitive. In this paper, we continue the classification process, and we classify the family of 2-geodesic transitive graphs of valency $3p$ where p is an odd prime.

A subgraph X of Γ is an *induced subgraph* if two vertices of X are adjacent in X if and only if they are adjacent in Γ. When $U \subseteq V(\Gamma)$, we denote by $[U]$ the subgraph of Γ induced by U. The action of G on a set Ω is said to be *primitive* if G has no non-trivial G-invariant partitions. There is a remarkable classification of finite primitive permutation groups mainly due to M. O'Nan and L. Scott, called the *O'Nan-Scott Theorem for primitive permutation groups.* They independently gave a

TABLE 1. locally primitive graphs

p	$[\Gamma(u)]$	X
3	$H(2,3)$	HA
3	$K_3 \times K_3$	AS
5	line graph of Petersen graph, $\overline{T(6)}$, $T(6)$	AS
7	line graph of Heawood graph, $\overline{T(7)}$, $T(7)$	AS
19	$L_2(19)^6_{57}$(=Perkel graph), $L_2(19)^{20}_{57}$ and $L_2(19)^{30}_{57}$	AS

classification of finite primitive groups, and divided them into 8 distinct types, see [11, 17]. A graph Γ is said to be *arc transitive*, if its automorphism group is transitive on the arc set.

Our main result classifies 2-geodesic transitive graphs of girth 3 with valency $3p$ where p is an odd prime, particularly those which are not 2-arc transitive.

Theorem 1.1. *Let Γ be a 2-geodesic transitive graph of valency $3p$ where p is an odd prime. Let $u \in V(\Gamma)$ and $A = \mathrm{Aut}(\Gamma)$. Then one of the following statements holds.*

(1) *$[\Gamma(u)]$ is connected and is A_u-arc transitive. If A_u is primitive on $\Gamma(u)$ of type X, then p, $[\Gamma(u)]$ and X lie in Table 1; if A_u is imprimitive on $\Gamma(u)$, then $\Gamma \cong K_{(m+1)[n]}$ where $(m,n) = (3,p)$ or $(p,3)$.*

(2) *$[\Gamma(u)]$ is a connected diameter 2 graph and is not A_u-arc transitive, and there exist such graphs for each p.*

(3) *$[\Gamma(u)]$ is disconnected and $[\Gamma(u)] \cong mK_n$ where $(m,n) = (3,p)$ or $(p,3)$. Further, Γ is the point graph of a partial linear space of order $(m, n+1)$ with girth at least 8, and there exist such graphs for each p.*

(4) *Γ has girth at least 4 and is 2-arc transitive.*

Remark 1.2. (1) Note that all graphs in Theorem 1.1 (1)-(3) have girth 3. There is no conjunction between any two of (1), (2) and (3).

(2)Proposition 2.6 provides infinitely many examples for Theorem 1.1 (2); and Example 2.7 provides infinitely many examples for Theorem 1.1 (3).

(3) Graphs in Theorem 1.1 (4) have been studied extensively, see [1, 8, 10, 12, 15, 21].

2. Proof of Theorem 1.1

Lemma 2.1. ([2, p.5] or [3]) *Let Σ be a connected graph that is locally complete multipartite. Then Σ is either triangle-free or complete multipartite. In particular, if $[\Sigma(v)] = \mathrm{K}_{m-1[b]}$, then $\Sigma = \mathrm{K}_{m[b]}$, where $v \in V(\Sigma)$ and m, b are integers, $m \geq 2, b \geq 2$.*

In a connected graph Γ, the smallest integer n such that there is a path of length n from u to v is called the *distance* from u to v and is denoted by $d_\Gamma(u, v)$. The *diameter* of Γ is the maximum of $d_\Gamma(u, v)$ over all $u, v \in V(\Gamma)$. A graph Γ is said to be *locally 2-geodesic transitive* if, for each vertex u and for $i = 1, 2$, the stabilizer A_u is transitive on i-geodesics starting from u, where $A = \mathrm{Aut}(\Gamma)$. In particular, a 2-geodesic transitive graph is both locally 2-geodesic transitive and vertex transitive.

Lemma 2.2. *Let Γ be a non-complete locally 2-geodesic transitive graph of girth 3. Let $A = \mathrm{Aut}(\Gamma)$ and $u \in V(\Gamma)$. Suppose that Γ is locally connected and $[\Gamma(u)]$ is A_u-arc transitive. Then either Γ is locally primitive or $\Gamma \cong \mathrm{K}_{(m+1)[n]}$ for some $m, n \geq 2$.*

Proof. Since Γ is non-complete, locally 2-geodesic transitive and locally connected of girth 3, it follows from [4, Theorem 1.1] that $[\Gamma(u)]$ has diameter 2.

Let $v \in \Gamma(u)$. Suppose that A_u is not primitive on $\Gamma(u)$ and Δ is a nontrivial block containing v. Since $[\Gamma(u)]$ is A_u-arc transitive, it follows that each nontrivial block contains no edges of $[\Gamma(u)]$, and hence there exists $v' \in \Delta$ such that v, v' are not adjacent, as Δ has at least 2 vertices. Then (v, u, v') is a 2-geodesic. Since Γ is locally 2-geodesic transitive, $A_{u,v}$ is transitive on $\Gamma(u) \cap \Gamma_2(v)$, and hence $\Gamma(u) \cap \Gamma_2(v) \subseteq \Delta$. Since $A_{u,v}$ is also transitive on $\Gamma(u) \cap \Gamma(v)$, $(\Gamma(u) \cap \Gamma(v)) \cap \Delta = \emptyset$. Thus $\Delta = \{v\} \cup (\Gamma(u) \cap \Gamma_2(v))$. Since any two vertices of Δ are not adjacent, it follows that $[\Gamma(u)]$ is an antipodal graph of diameter 2. Suppose $|\Gamma(u)| = s$ and $|\Delta| = n$. Then $s = mn$ for some $m \geq 2$. Further, A_u has m nontrivial blocks in $\Gamma(u)$. Thus $[\Gamma(u)] \cong \mathrm{K}_{m[n]}$. Finally, by Lemma 2.1, $\Gamma \cong \mathrm{K}_{(m+1)[n]}$. □

Lemma 2.3. *Let Ω be a set of size $3p$ where $3 \leq p$ is a prime. Let $G \leq \mathrm{Sym}(\Omega)$. If G is primitive on Ω, then the action type is HA or AS. Further, if $3 < p$, then the action type is AS.*

Proof. Suppose that G is primitive on Ω. If $p = 3$, then the primitive action type is HA or AS. Assume that $p \neq 3$. Then $3p$ is not a power of a number. Thus G is not primitive type of HA, HS, HC, SD, CD, PA or TW, and so G is primitive of type AS (or see [16]). □

A graph Γ is said to be *distance transitive* if $\mathrm{Aut}(\Gamma)$ is transitive on the ordered pairs of vertices at any given distance.

Lemma 2.4. *Let Γ be a 2-geodesic transitive graph of girth 3 with $3p$ vertices where $3 \leq p$. Let u be a vertex. Suppose that $[\Gamma(u)]$ is connected, A_u-arc transitive and A_u is primitive on $\Gamma(u)$ of type X. Then p, $[\Gamma(u)]$ and X are in Table 1.*

Proof. Since Γ is 2-geodesic transitive and $[\Gamma(u)]$ is connected, it follows from [4, Theorem 1.1] that $[\Gamma(u)]$ has diameter 2 and also A_u is transitive on nonadjacent vertex pairs of $[\Gamma(u)]$. Since $[\Gamma(u)]$ is A_u-arc transitive, it follows that $[\Gamma(u)]$ is distance transitive.

Suppose that A_u is primitive on $\Gamma(u)$ of type X. Then by Lemma 2.3, if $p = 3$, then X is HA or AS; if $p > 3$, then X is AS.

First, assume that $p = 3$. Then $[\Gamma(u)]$ has 9 vertices. Since Γ is non-complete, $[\Gamma(u)]$ is non-complete, and so $[\Gamma(u)]$ has valency $r \in \{2, \ldots, 7\}$. If $[\Gamma(u)]$ has valency 2, then by [5, Corollary 1.4], Γ is either the Octahedron or the Icosahedron, and neither has valency 9, a contradiction. Hence $[\Gamma(u)]$ has valency $r \in \{3, \ldots, 7\}$. Since $[\Gamma(u)]$ is distance transitive of diameter 2, we inspect candidates in [2, p.221-223], and $\mathrm{H}(2,3)$, $J(6,3)$ are the only two such graphs. In particular, if $[\Gamma(u)]$ is $\mathrm{H}(2,3)$, then X is HA; if $[\Gamma(u)]$ is $J(6,3)$, then X is AS.

Second, assume that $p > 3$. Note that arc transitive graphs of order $3p$ were classified by Wang and Xu [20], whenever $p >$

3 is a prime. Since $[\Gamma(u)]$ is connected, A_u-arc transitive and A_u is primitive on $\Gamma(u)$, it follows from [20, Theorem 2] that $[\Gamma(u)]$ is one of the following graphs: T_6, T_7, T_6^c, T_7^c, $L_3(2)_{21}^4$, $L_3(2)_{21}^8$, $L_3(2)_{21}^{8'}$, $L_2(19)_{57}^6$, $L_2(19)_{57}^{20}$ and $L_2(19)_{57}^{30}$. Since $[\Gamma(u)]$ is distance transitive, $[\Gamma(u)]$ lies in [2, p.221-225], and we inspect the candidates, p and Γ are in Table 1. □

Let $\Omega = \{1, 2, \ldots, n\}$ where $n \geq 3$, and let $1 \leq k \leq [\frac{n}{2}]$ where $[\frac{n}{2}]$ is the integer part of $\frac{n}{2}$. Then the *Johnson graph* $J(n, k)$ is the graph whose vertex set is the set of all k-subsets of Ω, and two vertices u and v are adjacent if and only if $|u \cap v| = k-1$. In particular, $J(n, 2)$ is called a *triangular graph*, denoted by $T(n)$. The *Cartesian product* $\Gamma_1 \square \Gamma_2$ of two graphs Γ_1 and Γ_2 is the graph with vertex set $V(\Gamma_1) \times V(\Gamma_2)$, and two vertices (v_1, v_2) and (u_1, u_2) are adjacent if and only if $v_1 = u_1$ and v_2, u_2 are adjacent in Γ_2 or $v_2 = u_2$ and v_1, u_1 are adjacent in Γ_1.

Lemma 2.5. *Let* $\Gamma = J(n, k)$ *where* $2 \leq k \leq [\frac{n}{2}]$ *and* $n \geq 4$. *Then for each vertex* v, $[\Gamma(v)] \cong \mathrm{K}_{n-k} \square \mathrm{K}_k$ *is connected of diameter* 2.

Proof. Let $u = \{1, \ldots, k\}$. Then $\Gamma(u) = \{u \setminus \{i\} \cup \{j\} | i \in \{1, \ldots, k\}, j \in \{k+1, \ldots, n\}\}$. For two vertices $v_1 = (u \setminus \{i_1\}) \cup \{j_1\}$ and $v_2 = (u \setminus \{i_2\}) \cup \{j_2\}$ of $\Gamma(u)$, v_1, v_2 are adjacent if and only if either $i_1 = i_2$ and $j_1, j_2 \in \{k+1, \ldots, n\}$, or $j_1 = j_2$ and $i_1, i_2 \in \{1, \ldots, k\}$. Thus the map taking $(u \setminus \{i\}) \cup \{j\}$ to (i, j) defines an isomorphism from $[\Gamma(u)]$ to $[\Delta] \square [\Delta']$ where $\Delta = \{(u \setminus \{1\}) \cup \{j\} | j \in \{k+1, \ldots, n\}\}$, $\Delta' = \{(u \setminus \{i\}) \cup \{n\} | i \in \{1, \ldots, k\}\}$, and $[\Delta] \cong \mathrm{K}_{n-k}$, $[\Delta'] \cong \mathrm{K}_k$. Since Γ is vertex transitive, it follows that $[\Gamma(v)] \cong \mathrm{K}_{n-k} \square \mathrm{K}_k$ is connected of diameter 2 for every vertex v. □

A transitive permutation group G is said to be *quasiprimitive*, if every nontrivial normal subgroup of G is transitive. In particular, every primitive permutation group is quasiprimitive, but the converse is not true. For knowledge of quasiprimitive permutation groups, see [13] and [14].

Now we show that Johnson graphs are 2-geodesic transitive but not 2-arc transitive, and for each vertex u, $[\Gamma(u)]$ is connected, and is not A_u-arc transitive.

Proposition 2.6. *Let* $\Gamma = J(n,k)$.

(1) *Assume that* $2 \leq k < \frac{n}{2}$ *and* $5 \leq n$. *Then* $\mathrm{Aut}(\Gamma)$ *acts primitively of type AS on* $V(\Gamma)$, *and for each vertex* u, $[\Gamma(u)] \cong \mathrm{K}_{n-k}\Box\mathrm{K}_k$ *is vertex transitive but not arc transitive.*

(2) *Assume that* $k = \frac{n}{2} \geq 3$. *Then* Γ *is an antipodal graph with fibres of size* 2, $\mathrm{Aut}(\Gamma)$ *is not quasiprimitive on* $V(\Gamma)$, *and for each vertex* u, $[\Gamma(u)] \cong \mathrm{K}_k\Box\mathrm{K}_k$ *is arc transitive. In particular,* $G \in \{A_n, S_n\}$ *acts quasiprimitively but not primitively on* $V(\Gamma)$.

Proof. (1) Since $n \neq 2k$, it follows that $A := \mathrm{Aut}(\Gamma) \cong S_n$. Hence for each vertex u, $A_u \cong S_k \times S_{n-k}$ is a maximal subgroup of A. Since A is transitive on $V(\Gamma)$ and $n \geq 5$, it follows that A is primitive of type AS on $V(\Gamma)$.

By Lemma 2.5, for each vertex u, $[\Gamma(u)] \cong \mathrm{K}_{n-k}\Box\mathrm{K}_k$. Since $n-k \neq k$, it follows that $[\Gamma(u)]$ is vertex transitive but not arc transitive.

(2) For $u \in V(\Gamma)$, let $\overline{u} = \{1,\ldots,n\} \setminus u$, the complement of u in $\{1,\ldots,n\}$. Thus $d_\Gamma(u,\overline{u}) = k$ and in fact $\Gamma_k(u) = \{\overline{u}\}$, by $(J*)$. Thus Γ is antipodal with fibres of size 2 forming an imprimitivity system for $A := \mathrm{Aut}(\Gamma)$ in $V(\Gamma)$.

Now $A \cong S_n \times Z_2$ and Z_2 is not transitive on $V(\Gamma)$. Hence A is not quasiprimitive on $V(\Gamma)$. However, as $n \geq 6$, A_n is a simple group, and so each subgroup $G \in \{A_n, S_n\}$ of A acts quasiprimitively but not primitively on $V(\Gamma)$.

Finally, by Lemma 2.5, $[\Gamma(u)] \cong \mathrm{K}_k\Box\mathrm{K}_k$, and $A_u \cong S_k \wr S_2$ acts arc transitively on it. □

Example 2.7. Let $\Gamma = \mathrm{H}(d,n)$ where $(d,n) = (3,p+1)$ or $(p,4)$. Then Γ is locally isomorphic to $d\mathrm{K}_{n-1}$, and by [6, Proposition 2.2], Γ is 2-geodesic transitive.

Proof of Theorem 1.1. Let Γ be a 2-geodesic transitive graph of valency $3p$ where p is an odd prime. If Γ has girth at least 4, then each 2-arc is a 2-geodesic, and so 2-arc transitive, (4) holds. In the following, we assume that Γ has girth 3.

Let $u \in V(\Gamma)$ and $A = \mathrm{Aut}(\Gamma)$. Since Γ is 2-geodesic transitive, it follows from [4, Theorem 1.1] that, either

(a) $[\Gamma(u)]$ is connected of diameter 2, and the induced action of A_u on $\Gamma(u)$ is transitive on vertices and on pairs of non-adjacent vertices; or

(b) $[\Gamma(u)] \cong m\mathrm{K}_n$ for some integers $m \geq 2, n \geq 2$.

First, assume that $[\Gamma(u)]$ is connected and is also A_u-arc transitive. Then by Lemmas 2.2 and 2.4, (1) holds. Second, assume that $[\Gamma(u)]$ is connected but not A_u-arc transitive. By [6, Proposition 2.1], the graph $J(n,k)$ is 2-geodesic transitive, where $2 \leq k \leq [\frac{n}{2}]$ and $n \geq 4$. Further, it follows from Proposition 2.6 that Johnson graphs provide infinitely many such examples, and (2) holds. Finally, assume that $[\Gamma(u)] \cong m\mathrm{K}_n$ for some integers $m \geq 2, n \geq 2$. Then $(m,n) = (3,p)$ or $(p,3)$, and by [4, Theorem 1.2], $[\Gamma(u)]$ is the point graph of a partial linear space of order $(m, n+1)$ with girth at least 8. By Example 2.7, there exist such graphs for each p. □

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