# Lattices associated with Hamming graphs

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#### Abstract

Hamming graph H(n,k) has as vertex set all words of length nwith symbols taken from a set of k elements. Suppose L denotes the set  $\bigcup_{l=0}^{n+1} \Omega_l$ , with  $\Omega_l = \{\sum_{i \in I_1} e_i^1 + \sum_{i \in I_2} e_i^2 + \dots + \sum_{i \in I_k} e_i^k \mid I_j \cap I_{j'} = \emptyset (j \neq j'), |\bigcup_{j=1}^k I_j| = l\}$  for  $0 \leq l \leq n$  and  $\Omega_{n+1} := \{\hat{1}\}$ . For any two element  $x, y \in L$ , define  $x \leq y$  if and only if  $y = \hat{1}$  or  $I_j^x \subseteq I_j^y$  for  $1 \leq j \leq k$ . Then L is a lattice, denoted by  $L_O$ . Reversing the above partial order, we obtain the dual of  $L_O$ , denoted by  $L_R$ . This article discusses their geometric properties, and computes their characteristic polynomials.

**AMS** classification: 05B35; 20G40 **Keywords:** Lattice; Geometric lattice; Hamming graph

#### **1** Introduction

Hamming graph H(n, k) has as vertex set all words of length n with symbols taken from a set of k elements. We will take as our set of k elements the set  $\{a_1, a_2, \dots, a_k\}$ .

Let  $e_i^j$  be the vector with n coordinates that has a  $a_j$  in position iand 0 elsewhere. Then, each word in H(n,k) is simply a sum of some  $e_i^1, e_i^2, \dots, e_i^k$   $(i \in I_j \text{ for } 0 \leq j \leq k)$ , with the only restrictions on  $I_j$ 's that  $I_j \cap I_{j'} = \emptyset$   $(j \neq j')$ , and  $\bigcup_{j=1}^k I_j = \{1, 2, \dots, n\}$ .

$$\begin{split} & e_i, e_i, \dots, e_i \ (i \in I_j \text{ for } i = j = j) = I_i \\ & I_j \cap I_{j'} = \emptyset \ (j \neq j'), \text{ and } \bigcup_{j=1}^k I_j = \{1, 2, \dots, n\}. \\ & \text{For } 0 \le l \le n \text{ we set } \Omega_l = \{\sum_{i \in I_1} e_i^1 + \sum_{i \in I_2} e_i^2 + \dots + \sum_{i \in I_k} e_i^k \mid I_j \cap I_{j'} = \emptyset \ (j \neq j'), |\bigcup_{j=1}^k I_j| = l\}. \text{ Given any } x \in \Omega_l, \text{ we represent } x = (I_1^x, I_2^x, \dots, I_k^x) \text{ where } x = \sum_{i \in I_1^x} e_i^1 + \sum_{i \in I_2^x} e_i^2 + \dots + \sum_{i \in I_k^x} e_i^k, I_j^x \cap I_{j'}^x = \emptyset \ (j \neq j') \text{ and } |\bigcup_{j=1}^k I_j^x| = l. \end{split}$$

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Notice that  $\Omega_0 := \{(0, 0, \dots, 0)\}$  and we add a dummy element  $\hat{1}$  above

all other elements, defining  $\Omega_{n+1} := \{\hat{1}\}$ , that is  $x \leq \hat{1}, \forall x \in \bigcup_{l=0}^{n} \Omega_l$ . Suppose L denotes the set  $\bigcup_{l=0}^{n+1} \Omega_l$ . For any two elements  $x, y \in L$ , define  $x \leq y$  if and only if  $y = \hat{1}$  or  $I_j^x \subseteq I_j^y$  for  $0 \leq j \leq k$ . Then L is a finite poset, denoted by  $L_O$ . For any two elements  $x, y \in L$ , define  $x \leq y$  if and only if  $y = (0, \dots, 0)$  or  $I_j^y \subseteq I_j^x$  for  $0 \le j \le k$ . Then L is a finite poset, denoted by  $L_R$ .

For any two elements  $x, y \in L_O$ ,

$$\begin{aligned} x \wedge y &= (I_1^x \cap I_1^y, I_2^x \cap I_2^y, \cdots, I_k^x \cap I_k^y), \\ x \vee y &= \begin{cases} (I_1^x \cup I_1^y, I_2^x \cup I_2^y, \cdots, I_k^x \cup I_k^y), & \text{if } (I_j^x \cup I_j^y) \cap (I_{j'}^x \cup I_{j'}^y) = \emptyset(j \neq j'), \\ \hat{1}, & \text{otherwise.} \end{cases} \end{aligned}$$

Similarly, for any two elements  $x, y \in L_R$ ,

$$\begin{aligned} x \lor y &= (I_1^x \cap I_1^y, I_2^x \cap I_2^y, \cdots, I_k^x \cap I_k^y), \\ x \land y &= \left\{ \begin{array}{ll} (I_1^x \cup I_1^y, I_2^x \cup I_2^y, \cdots, I_k^x \cup I_k^y), & \text{if } (I_j^x \cup I_j^y) \cap (I_{j'}^x \cup I_{j'}^y) = \emptyset(j \neq j'), \\ \hat{1}, & \text{otherwise.} \end{array} \right. \end{aligned}$$

Therefore, both  $L_O$  and  $L_R$  are finite lattices.

Y. Huo, Y. Liu and Z. Wan ([3, 4, 5, 6, 7]) constructed lattices from orbits of subspaces under finite classical groups. K. Wang and Y. Feng constructed lattices from orbits of flats under affine groups. K. Wang and Z. Li [11] constructed lattices from vector spaces over a finite field. In this paper, we construct two families of lattices from Hamming graphs, compute their characteristic polynomials and discuss their geometric properties.

#### **Preliminaries** $\mathbf{2}$

We recall some terminologies and definitions about finite posets and lattices. For more theory about finite posets and lattices, we would like to refer readers to [1, 9].

Let P be a poset with partial order  $\leq$ . As usual, we write a < bwhenever  $a \leq b$  and  $a \neq b$ . For any two elements  $a, b \in P$ , we say b covers a, denoted by a < b, if a < b and there exists no any  $c \in P$  such that a < c < b. Let P be a finite poset with the minimum element, denoted by 0. By a rank function on P, we mean a function r from P to the set of all the integers such that r(0) = 0 and r(a) = r(b) - 1 whenever a < b. Observe the rank function of P is unique if it exists. Let P be a finite poset with 0 and 1. The polynomial

$$\chi(P, x) = \sum_{a \in P} \mu(0, a) x^{r(1) - r(a)}$$

is called the *characteristic polynomial* of P, where r is the rank function of P.

A poset L is said to be a *lattice* if both  $a \lor b := \sup\{a, b\}$  and  $a \land b := \inf\{a, b\}$  exist for any two elements  $a, b \in L$ . Let L be a finite lattice with 0. By an *atom* of L, we mean an element of L covering 0. We say L is *atomic* if any element in  $L \setminus \{0\}$  is a union of atoms. A finite atomic lattice L is said to be a *geometric lattice* if L admits a rank function r satisfying

$$r(a \wedge b) + r(a \vee b) \le r(a) + r(b), \forall a, b \in L.$$

## **3** The lattice $L_O$

The lattice  $L_O$  has the minimum element  $\hat{0} = (0, 0, \dots, 0)$ , and the maximum element  $\hat{1}$ . The set of all the atoms of  $L_O$  is  $\Omega_1$ .

**Theorem 3.1** The lattice  $L_O$  has the following properties:

- (i)  $L_O$  is a finite atomic lattice, that is every element of the lattice is a join of atoms.
- (ii)  $\forall u, w \in L_O$  such that  $u \lor w \neq \hat{1} \Rightarrow r(u \land w) + r(u \lor w) = r(u) + r(w)$ .

*Proof.* For any  $z \in L_O$ , define

$$r(z) = \begin{cases} n+1, & \text{if } z = \hat{1}, \\ |I_1^z| + |I_2^z| + \dots + |I_k^z|, & \text{otherwise} \end{cases}$$

Then r is the rank function of  $L_O$ .

(i)  $\hat{1} = e_1^1 \vee e_1^2$  and an element  $z \neq \hat{1}$  of the lattice is of the form

$$z = \sum_{i \in I_1^z} e_i^1 + \sum_{i \in I_2^z} e_i^2 + \dots + \sum_{i \in I_k^z} e_i^k, I_j^z \cap I_{j'}^z = \emptyset(j \neq j'),$$

so  $z = (\bigvee_{i \in I_1^z} e_i^1) \bigvee (\bigvee_{i \in I_2^z} e_i^2) \bigvee \cdots \bigvee (\bigvee_{i \in I_k^z} e_i^k)$  is a join of atoms.

(ii)  $r(z) = |I_1^z| + |I_2^z| + \dots + |I_k^z|$ , so the formula is true because of the inclusion-exclusion formula of sets.

**Lemma 3.2** The Möbius function of  $L_O$  is

$$\mu(x,y) = \begin{cases} (-1)^{r(y)-r(x)}, & \text{if } x \le y \ne \hat{1} \text{ or } x = y = \hat{1}, \\ -(1-k)^{n-r(x)}, & \text{if } x < y = \hat{1}, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* The Möbius function of  $L_O$  is

$$\mu(x,y) = \begin{cases} (-1)^{r(y)-r(x)}, & \text{if } x \le y \ne \hat{1} \text{ or } x = y = \hat{1}, \\ -\sum_{x \le z < y} \mu(x,z), & \text{if } x < y = \hat{1}, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} & -\sum_{\substack{\hat{0} \leq z < \hat{1} \\ = -\sum_{i=0}^{n} k^{i} C_{n}^{i} (-1)^{i} \\ = -(1-k)^{n} \end{aligned}$$

and

$$\begin{array}{rl} & -\sum\limits_{\hat{0}\neq x\leq z<\hat{1}}\mu(x,z)\\ = & -\sum\limits_{i=0}^{n-r(x)}k^{i}C_{n-r(x)}^{i}(-1)^{i}\\ = & -(1-k)^{n-r(x)}. \end{array}$$

Hence the desired result follows.

**Theorem 3.3** The characteristic polynomial of  $L_O$  is

$$\chi(L_O, x) = -(1-k)^n + x(x-k)^n.$$

Proof. By Lemma 3.2, we obtain

$$\begin{aligned} \chi(L_O, x) &= \sum_{\substack{\hat{0} \le y \le \hat{1} \\ = \mu(\hat{0}, \hat{1}) + \sum_{\substack{\hat{0} \le y < \hat{1} \\ i = 0}} \mu(\hat{0}, \hat{1}) + \sum_{\substack{\hat{0} \le y < \hat{1} \\ i = 0}} \mu(\hat{0}, y) x^{n+1-r(y)} \\ &= -(1-k)^n + \sum_{\substack{i=0 \\ i = 0}}^n k^i C_n^i (-1)^i x^{n+1-i} \\ &= -(1-k)^n + x(x-k)^n, \end{aligned}$$

as desired.

# 4 The lattice $L_R$

The lattice  $L_R$  has the minimum element  $\hat{1}$ , and the maximum element  $\hat{0} = (0, 0, \dots, 0)$ . The set of all the atoms of  $L_R$  is  $\Omega_n$ .

**Theorem 4.1** The lattice  $L_R$  has the following properties:

(i)  $L_R$  is a finite atomic lattice, that is every element of the lattice is a join of atoms.

(ii)  $\forall u, w \in L_R \text{ such that } u \lor w \neq \hat{0} \Rightarrow r(u \land w) + r(u \lor w) = r(u) + r(w).$ 

*Proof.* For any  $z \in L_R$ , define

$$r(z) = \begin{cases} n+1, & \text{if } z = \hat{0}, \\ n+1 - (|I_1^z| + |I_2^z| + \dots + |I_k^z|), & \text{otherwise}, \\ 0, & \text{if } z = \hat{1}, \end{cases}$$

Then r is the rank function of  $L_R$ .

(i)  $\hat{0} = e_1^1 \vee e_1^2$  and an element  $z \neq \hat{0}$  of the lattice is of the form

$$z = \sum_{i \in I_1^z} e_i^1 + \sum_{i \in I_2^z} e_i^2 + \dots + \sum_{i \in I_k^z} e_i^k, |I_1^z| + |I_2^z| + \dots + |I_k^z| = l,$$

there are  $x, y \in \Omega_n$ ,

$$\begin{aligned} x &= \sum_{i \in I_1^x} e_i^1 + \sum_{i \in I_2^y} e_i^2 + \dots + \sum_{i \in I_k^x} e_i^k, \ |I_1^x| + |I_2^x| + \dots + |I_k^x| = n, \\ y &= \sum_{i \in I_1^y} e_i^1 + \sum_{i \in I_2^y} e_i^2 + \dots + \sum_{i \in I_k^y} e_i^k, \ |I_1^y| + |I_2^y| + \dots + |I_k^y| = n, \end{aligned}$$

with

$$I_1^z = I_1^x \cap I_1^y, I_2^z = I_2^x \cap I_2^y, \cdots, I_k^z = I_k^x \cap I_k^y,$$

then  $z = x \lor y$  is a join of atoms.

(ii)  $r(z) = n + 1 - (|I_1^z| + |I_2^z| + \dots + |I_k^z|)$ , so the formula is true because of the inclusion-exclusion formula of sets.

**Lemma 4.2** The Möbius function of  $L_R$  is

$$\mu(x,y) = \begin{cases} (-1)^{r(x)-r(y)}, & \text{if } \hat{1} \neq x \le y \text{ or } x = y = \hat{1}, \\ -(1-k)^{r(y)-1}, & \text{if } \hat{1} = x < y, \\ 0, & \text{otherwise.} \end{cases}$$

*Proof.* The Möbius function of  $L_R$  is

$$\mu(x,y) = \begin{cases} (-1)^{r(x)-r(y)}, & \text{if } \hat{1} \neq x \le y \text{ or } x = y = \hat{1}, \\ -\sum_{x < z \le y} \mu(z,y), & \text{if } \hat{1} = x < y, \\ 0, & \text{otherwise.} \end{cases}$$

We have

$$\begin{aligned} & -\sum_{\substack{\hat{1} < z \leq y \\ z = -\sum_{i=0}^{r(y)-1} k^i C^i_{r(y)-1}(-1)^i \\ = & -(1-k)^{r(y)-1}. \end{aligned}$$

Hence the desired result follows.

**Theorem 4.3** The characteristic polynomial of  $L_R$  is

$$\chi(L_R, x) = x^{n+1} - (1 - k + kx)^n.$$

*Proof.* By Lemma 4.2, we obtain

$$\begin{array}{rcl} & \chi(L_R, x) \\ = & \sum\limits_{\hat{1} \le y \le \hat{0}} \mu(\hat{1}, y) x^{n+1-r(y)} \\ = & \mu(\hat{1}, \hat{1}) x^{n+1} + \sum\limits_{\hat{1} < y \le \hat{0}} \mu(\hat{1}, y) x^{n+1-r(y)} \\ = & x^{n+1} - \sum\limits_{i=0}^{n} k^i C_n^i (1-k)^{n-i} x^i \\ = & x^{n+1} - (1-k+kx)^n, \end{array}$$

as desired.

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