ON THE AUTOMORPHISM GROUPS OF REGULAR HYPERSTARS AND FOLDED HYPERSTARS

S. MORTEZA MIRAFZAL

ABSTRACT. The hyper-star graph $HS(n,k)$ is defined as follows : its vertex-set is the set of $\{0,1\}$-sequences of length n with weight k, where the weight of a sequence v is the number of $1's$ in v, and two vertices are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0, or 0 with 1) in another position. In this paper, we will find the automorphism groups of regular hyper-star and folded hyper-star graphs. Then, we will show that, only the graphs $HS(4,2)$ and $FHS(4,2)$ are Cayley graphs.

Keywords : Vertex transitive graph; Permutation group; Symmetric graph; Cayley graph

AMS Subject Classifications: 05C25; 94C15

1. INTRODUCTION AND PRELIMINARIES

An interconnection network can be represented as an undirected graph where a processor is represented as a vertex and a communication channel between processors as an edge between corresponding vertices. Measures of the desirable properties for interconnection networks include degree, connectivity, scalability, diameter, fault tolerance, and symmetry [1]. For example in [4, 6] have been found the symmetries of two important classes of graphs. The main aim of this paper is to study the symmetries of a class of graphs that are useful in some aspects for designing some interconnection networks. First major class of interconnection networks is the classical n-cubes. Star graphs were introduced by [1] as a competitive model to the n-cubes. Both the n-cubes and Star graphs have been studied and many of the properties are known and star graphs have proven to be superior to the n-cubes. The hyper-star graphs were introduced in [8] as competitive model to both n-cubes and star graphs. Some of the structural and topological properties of hyper-star graphs have been studied in [3, 7]. For all the terminology and notation not defined here, we follow [2, 5, 10]. Let $n > 2$, the hyper-star graph $HS(n,k)$ where, $1 \leq k \leq n-1$, is defined in [8] as follows : its vertex-set is the set of $\{0,1\}$-sequences of length n with weight k, where the weight of the sequence v is the number of $1's$ in v, and two vertices are adjacent if and only if one can be obtained from the other by exchanging the first symbol with a different symbol (1 with 0 or 0 with 1) in another position. Formally, if we denote by $V(HS(n,k))$ and $E(HS(n,k))$ the vertex-set and edge-set of $HS(n,k)$ respectively, then

$V = V(HS(n,k)) = \{x_1x_2 \cdots x_n \mid x_i \in \{0,1\}, \sum_{j=1}^{n} x_j = k\}$

$E = E(HS(n,k)) = \{\{u,v\} \mid u = x_1x_2 \cdots x_n, v = x_ix_2 \cdots x_{i-1}x_1x_{i+1} \cdots x_n, x_1 = x_i^c\}$, where x^c is the complement of x ($0^c = 1$ and $1^c = 0$). It is clear that the degree of a vertex v of $HS(n,k)$ is, $n-k$ if $1 \in v$, or is k if $1 \notin v$. So $HS(n,k)$ is regular if and only if $n = 2k$.

Let $X = \{1, 2, ..., n\}$ and X_k be the family of subsets of X with k elements. Let $S(n,k)$ be the graph with vertex-set X_k and two vertices $v = \{x_1, \cdots, x_k\}$ and $w = \{y_1, \cdots, y_k\}$ are adjacent if and only if $\mid v \cap w \mid = k-1$ and, 1 belongs to one, and only one, of the vertices v and w, in other words w is obtained from v by replacing an element $y \in X - v$ with 1, if $1 \in v$, and replacing $x \in v$ by 1 if, $1 \notin v$. Let A be a subset of X, then the characteristic function of A is the function $\chi_A : X \longrightarrow \{0,1\}$ such that $\chi_A(x) = 1$, if and only if $x \in A$. Thus $A \longmapsto \chi_A$ is a bijection between the family of subsets of X and the set of sequences of $\{0,1\}$ of length n. The graphs $\Gamma_1 = (V_1, E_1)$ and $\Gamma_2 = (V_2, E_2)$ are called isomorphic, if there is a bijection $\alpha : V_1 \longrightarrow V_2$ such that, $\{a,b\} \in E_1$ if and only if $\{\alpha(a), \alpha(b)\} \in E_2$ for all $a, b \in V_1$. in such a case the bijection α is called an isomorphism. Now it is an easy task to show that the graphs $HS(n,k)$ and $S(n,k)$ are isomorphic, in fact the correspondence $A \longmapsto \chi_A$ is an isomorphism between $S(n,k)$ and $HS(n,k)$, and for this reason, from now on, we work with $S(n,k)$ and we denote it by $HS(n,k)$. The following figure shows the graph $HS(6,3)$, where the set $\{x,y,z\}$ is denoted by xyz.

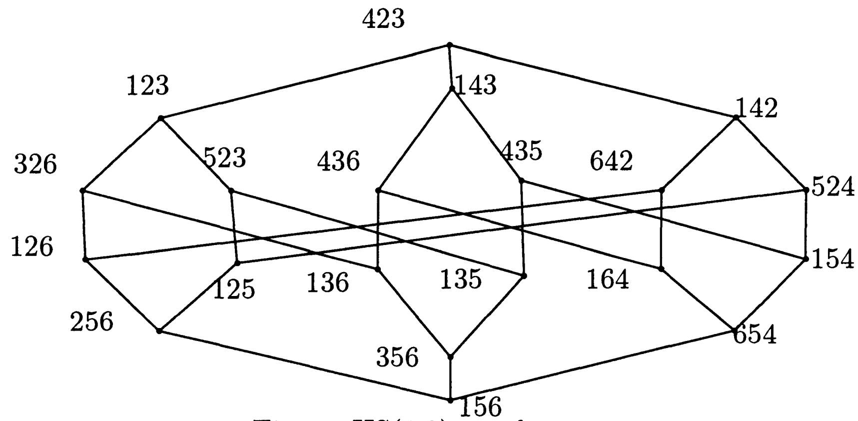


Fig. 1. HS(6,3) graph

An automorphism of a graph Γ is an isomorphism of Γ with itself. The set of all automorphisms of Γ, with the operation of composition of functions, is a group, called the automorphism group of Γ and denoted by $Aut(\Gamma)$. A permutation of a set is a bijection of it with itself. The group of all permutations of a set V is denoted by $Sym(V)$, or just $Sym(n)$ when $\mid V \mid= n$. A permutation group G on V is a subgroup of $Sym(V)$. In this case we say that G acts on V. If Γ is a graph with vertex-set V, then we can view each automorphism as a permutation of V, and so $Aut(\Gamma)$ is a permutation group. Let G act on V, we say that G is transitive (or G acts transitively on V) if there is just one orbit. This means that given any two elements u and v of V, there is an element β of G such that $\beta(u) = v$.

The graph Γ is called vertex transitive if $Aut(\Gamma)$ acts transitively on $V(\Gamma)$.The action of $Aut(\Gamma)$ on $V(\Gamma)$ induces an action on $E(\Gamma)$ by the rule $\beta\{x,y\} = \{\beta(x), \beta(y)\}, \beta \in Aut(\Gamma)$, and Γ is called edge transitive if this action is transitive.The graph Γ is called symmetric, if for all vertices $u, v, x, y,$ of Γ such that u and v are adjacent, and x and y are adjacent, there is an automorphism α such that $\alpha(u) = x, and, \alpha(v) = y$. It is clear that a symmetric graph is vertex transitive and edge transitive.

For $v \in V(\Gamma)$ and $G = Aut(\Gamma)$, the stabilizer subgroup G_v is the subgroup of G containing all automorphisms which fix v. In the vertex transitive case all stabilizer subgroups G_v are conjugate in G, and consequently isomorphic, in this case, the index of G_v in G is given by the equation, $|G : G_v| = \frac{|G|}{|G_v|} = |V(\Gamma)|$. If each stabilizer G_v is the identity group, then every element of G, except the identity, does not fix any vertex, and we say that G acts semiregularly on V. We say that G acts regularly on V if and only if G acts transitively and semiregularly on V and in this case we have $| V |=| G |$.

Let G be any abstract finite group with identity 1, and suppose that Ω is a set of generators of G, with the properties :

(i) $x \in \Omega \Longrightarrow x^{-1} \in \Omega; (ii) 1 \notin \Omega$;

The Cayley graph $\Gamma = \Gamma(G, \Omega)$ is the (simple) graph whose vertex-set and edge-set defined as follows :

$V(\Gamma) = G; E(\Gamma) = \{\{g, h\} \mid g^{-1}h \in \Omega\}$. It can be shown that a connected graph Γ is a cayley graph if and only if $Aut(\Gamma)$ contains a subgroup H, such that H acts regularly on $V(\Gamma)$ [2, 5].

The group G is called a semidirect product of N by Q, denoted by $G = N \rtimes Q$, if G contains subgroups N and Q such that, (i)$N \trianglelefteq G$ (N is a normal subgroup of G); (ii) $NQ = G$; (iii) $N \cap Q = 1$.

2. Main results

In the remaining of this section we assume that k is a fixed natural number, but arbitrarily chosen and $k > 2$ and $X = \{1, 2, ..., 2k\}$.

Lemma 2.1. *The graph $HS(2k, k)$ is a vertex transitive graph.*

Proof. In [8] it is proved that $HS(2k, k)$ is a vertex transitive graph and in [3] it is proved that this graph is edge transitive, but for the sake of consistency and, since our proof is independent of those and we need our proof in the sequel, we bring a proof. Let $V = V(HS(2k, k))$. The graph $HS(2k, k)$ is a regular bipartite graph of valency (regularity k), in fact if $P_1 = \{v \in V \mid 1 \in v\}$ and $P_2 = \{w \in V \mid 1 \notin w\}$ then, $\{P_1, P_2\}$ is a partition of V and every edge of $HS(2k, k)$ has a vertex in P_1 and a vertex in P_2 and $| P_1 |=| P_2 |$. Let α be a permutation of $Sym(X)$ such that α fixes the element 1. α induces a permutation $\tilde{\alpha}$ on V by the rule $\tilde{\alpha}(\{x_1, x_2, \cdots, x_k\}) = \{\alpha(x_1), \alpha(x_2), \cdots, \alpha(x_k)\}$. We have $| v \cap w | =$

$\mid \tilde{\alpha}(v) \cap \tilde{\alpha}(w) \mid$ and 1 is in one, and only one of the vertices of an edge, thus $\tilde{\alpha}$ is an automorphism of the graph $HS(2k,k)$. Note that if $v \in P_1$, then $\tilde{\alpha}(v) \in P_1$, thus $\tilde{\alpha}(P_1) = P_1$ and $\tilde{\alpha}(P_2) = P_2$. For any vertex v in V, let v^c be the complement of the set v in X. We define the mapping $\theta : V \longrightarrow V$ by the rule, $\theta(v) = v^c$, for every v in V. In fact θ is an automorphism of $HS(2k,k)$. Note that for any α in $Sym(X)$ that fixes 1 , $\tilde{\alpha} \neq \theta$. Now, let $v, w \in V$. Suppose $v, w \in P_1$ and $|v \cap w| = t$. Let $v = \{1, x_2, ..., x_t, y_1, ..., y_{k-t}\}$ and $w = \{1, x_2, ..., x_t, z_1, ..., z_{k-t}\}$. We define the permutation $\pi \in Sym(X)$ by the rule; $\pi(1) = 1, \pi(x_i) = x_i, \pi(y_j) = z_j$, and $\pi(u) = u, u \in X - (v \cup w)$. Thus, $\tilde{\pi}$ is an automorphism of $HS(2k,k)$ and $\tilde{\pi}(v) = w$. If $v, w \in P_2$ then, $\theta(v), \theta(w) \in P_1$, therefore there is an automorphism $\tilde{\pi}$ in $Aut(HS(2k,k))$ such that $\tilde{\pi}(\theta(v)) = \theta(w)$, thus $(\theta^{-1}\tilde{\pi}\theta)(v) = w$. Now, let $v \in P_1$ and $w \in P_2$, thus $\theta(w) \in P_1$ and there is an automorphism $\tilde{\pi} \in Aut(HS(2k,k))$ such that $\tilde{\pi}(v) = \theta(w)$, then we have $\theta^{-1}\tilde{\pi}(v) = w$.

□

For a graph Γ and $v \in V(\Gamma)$, let $N(v)$ be the set of vertices w of Γ such that w is adjacent to v. If $G = Aut(\Gamma)$, then G_v acts on $N(v)$, if we restrict the domains of the permutations $g \in G_v$ to $N(v)$. It is an easy task to show that a vertex transitive graph Γ is symmetric, if and only if, G_v acts transitively on the set $N(v)$ for any $v \in V(\Gamma)$. In the sequel θ is the automorphism of $HS(2k,k)$ which is defined in Lemma 2.1.

Theorem 2.2. *The graph $HS(2k,k)$ is a symmetric graph.*

Proof. Let $\Gamma = HS(2k,k)$ and $G = Aut(HS(2k,k))$. Since Γ is a vertex transitive graph, it is enough to show that G_v acts transitively on $N(v)$ for any $v \in V = V(\Gamma)$. Let $v \in P_1$, $v = \{1, x_2, ..., x_k\}$, thus $N(v) = \{\{y_i, x_2, ..., x_k\} \mid 1 \leq i \leq k\}$, where $X = \{1, x_2, ..., x_k, y_1, ..., y_k\}$. If $w_i, w_j \in N(v)$, $w_i = \{y_i, x_2, ..., x_k\}$, $w_j = \{y_j, x_2, ..., x_k\}$, then the transposition $\tau = (y_i y_j) \in Sym(X)$ is such that $\tilde{\tau}$ is in G_v and $\tilde{\tau}(w_i) = w_j$. Now let $v \in P_2$ and $u, w \in N(v)$, thus $\theta(v) = v^c \in P_1$ and $\theta(u), \theta(w) \in N(\theta(v)) = N(v^c)$. Therefore there is an automorphism $\pi \in G_{v^c}$ such that

$\pi(\theta(u)) = \theta(v)$. Thus, $(\theta^{-1}\pi\theta)(u) = w$ and since $\pi(\theta(v)) = \theta(v)$, we have $(\theta^{-1}\pi\theta)(v) = v$. □

Suppose Γ is a graph and $G = Aut(\Gamma)$. For a vertex v of Γ, let L_v be the set of all elements g of G_v such that g fixes each element of $N(v)$. Let $L_{vw} = L_v \cap L_w$.

Lemma 2.3. *Let Γ be a graph such that every vertex of it is of degree greater than one and $G = Aut(\Gamma)$. If v be a vertex of Γ of degree b, and w be an element of $N(v)$ with minimum degree m, then, $\mid G_v \mid \leq b!(m-1)!|L_{vw}|$.*

Proof. Let $Y = N(v)$ and $\Phi : G_v \longrightarrow Sym(Y)$ be defined by the rule, $\Phi(g) = g_{|Y}$ for any element g in G_v , where $g_{|Y}$ is the restriction of g to Y. In fact Φ is a group homomorphism and $ker(\Phi) = L_v$, thus G_v/L_v is isomorphic with a subgroup of $Sym(Y)$. Since, $\mid Y \mid = deg(v) = b$, therefore $\mid G_v \mid / \mid L_v \mid \leq b!$.

Now, $\mid G_v \mid \leq (b!) \mid L_v \mid$. If w is an element of $N(v)$ of degree l and $g \in L_v$, then g fixes $v \in N(w)$. Let $Z = N(w) - \{v\}$ and $\Psi : L_v \longrightarrow Sym(Z)$ be defined by $\Psi(h) = h_{|Z}$, for any element h in L_v. Then the kernel of the homomorphism Ψ is L_{vw} and since $\mid Z \mid = l-1$, thus $\mid L_v \mid \leq (l-1)! \mid L_{vw} \mid$. Now, we have $\mid G_v \mid \leq b!(l-1)! \mid L_{vw} \mid$. If w be an element in $N(v)$ of minimum degree m, then the result follows.

□

From the previous Lemma it follows that, if Γ is a regular graph of degree m, then for every edge $\{v, w\}$ of Γ we have $\mid G_v \mid \leq m!(m-1)! \mid L_{vw} \mid$.

Theorem 2.4. *The automorphism group of $HS(2k, k)$ is a semidirect product of N by Q, where N is isomorphic to $Sym(2k-1)$ and Q is isomorphic to Z_2, the cyclic group of order 2.*

Proof. If H be the subgroup of $Sym(X)$ that contains permutations which fix the element 1, then H is isomorphic with $Sym(2k-1)$. Then $f : H \longrightarrow Aut(HS(2k, k)) = G$, defined by $f(\alpha) = \tilde{\alpha}$, ($\tilde{\alpha}$ is defined in Lemma 2.1) is an injection. In fact, if $\alpha \neq 1$ be in $Sym(X)$ and $\alpha(1) = 1$, then there is an $x \in X$ such that $\alpha(x) \neq x$. Now, let T be a k-subset of X such that $x \in T$

and $\alpha(x) \notin T$. Then $\tilde{\alpha}(T) \neq T$ and hence $\tilde{\alpha} \neq 1$. It follows that the kernel of the homomorphism f is the identity group. Therefore, the subgroup $f(H) = N = \{\tilde{\alpha} \mid \alpha \in H\}$ is of order $(2k-1)!$. If Q be the cyclic subgroup of G generated by θ (θ is defined in Lemma 2.1), then $\mid Q \mid= 2$. Since, $\theta \notin N$, so $N \cap Q = 1$, thus for the set $NQ \subseteq G$ we have $\mid NQ \mid= \frac{|N||Q|}{1} = (2k-1)!(2)$, so we have $\mid G \mid\geq (2k-1)!(2)$. If we show that $\mid G \mid\leq (2k-1)!(2)$, then we must have $G = NQ$ and since the index of N in $NQ = G$ is 2, then N is a normal subgroup of G and the theorem will be proved. In the first step of the remaining proof, we assert that every 3-path in the graph $\Gamma = HS(2k, k)$ determines a unique 6-cycle in this graph. Let $P : v_1v_2v_3v_4$ be a 3-path in Γ. The path P has a form such as, $v_1 = \{y_1, x_2, x_3, ...x_k\}v_2 = \{1, x_2, x_3, ..., x_k\}v_3 = \{y_2, x_2, x_3, ..., x_k\}v_4 = \{y_2, 1, x_3, ..., x_k\}$. If C be a 6-cycle of Γ that contains P, then C has two adjacent vertices v_5 and v_6 such that v_5 is adjacent to v_4 and v_6 is adjacent to v_1. Thus v_5 has a form such as $v_5 = \{y_2, s, x_3, ..., x_k\}$ where, $s \in \{y_1, y_3, ..., y_k, x_1\}$ and v_6 has a form such as $v_6 = \{y_1, x_2, ..., x_{i-1}, 1, x_{i+1}, ..., x_k\}$. Since v_5 and v_6 are adjacent we must have $v_5 = \{y_2, y_1, x_3, ..., x_k\}$ and $v_6 = \{y_1, 1, x_3, ..., x_k\}$. Now the assertion is proved. In the second step we show that if $\{v, w\}$ be an edge of Γ, then $L_{v,w} = 1$. Let $g \in L_{v,w}$ and x be a vertex of Γ of distance 2 from v. If x is adjacent to w, then $g(x) = x$. Let x is not adjacent to w, so there is a vertex y adjacent to v such that vyx is a 2-path of Γ. If $C : xyvwtu$ be the unique 6-cycle that contains the 3-path $xyvw$, then $g(C)$ is the 6-cycle $g(x)yvwtg(u)$, so C and $g(C)$ contain the 3-path $yvwt$, thus $g(C) = C$. Therefore $g_{|V(C)}$ is an automorphism of 6-cycle C that fixes the 2-path wvy, thus g fixes all vertex of this cycle and we have $g(x) = x$. Now, since the graph Γ is connected , it follows that g fixes all the vertices of Γ, so $g = 1$ and $L_{v,w} = 1$.

The graph Γ is vertex transitive, thus for a vertex $v \in V = V(\Gamma)$ we have;

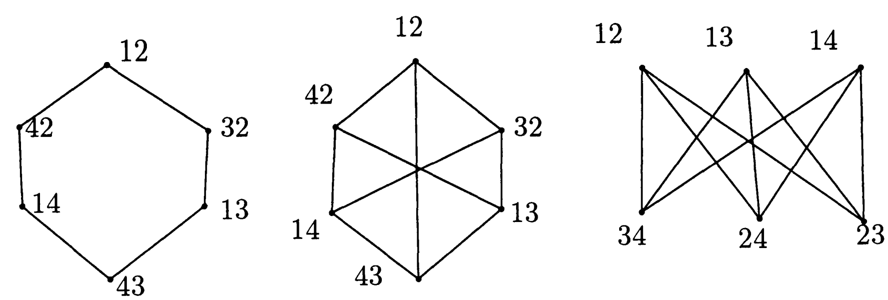
$$\mid G \mid=\mid V \mid\mid G_v \mid\leq \binom{2k}{k}(k!)(k-1)! = \frac{2k!}{k!k!}k!(k-1)! = \frac{2k!}{k} = (2k-1)!2$$

□

Remark: As we can see in the proof of Theorem 2.4, the graph $HS(2k,k)$ has 6-cycles and since this graph is bipartite , hence it has no 3-cycles and no 5-cycles. It is easy to show that this graph has no 4-cycles, so the girth of this graph is 6.

3. Folded hyper-star graphs

The folded hyper star-graph $FHS(2k,k)$ is the graph which its vertex-set is identical to the vertex-set of hyper-star graph $HS(2k,k)$, and with edge-set $E_2 = E_1 \cup \{\{v, v^c\} \mid v \in V_1\}$, where E_1 and V_1 are the edge-set and vertex-set of $HS(2k,k)$ respectively. It is clear that this graph is a regular bipartite graph of degree $k+1$. It is an easy task to show that the diameter of $FHS(2k,k)$ is k, whereas the diameter of $HS(2k,k)$ is $2k-1$ [8]. We will show that this graph is also vertex transitive, thus its edge connectivity is maximum, say $k+1$ [11] . Let v be a vertex of $FHS(2k,k)$. We can suppose that $v = \{1, x_2, ..., x_k\}$, then $N(v) = \{\{y_i, x_2, ..., x_k\}, 1 \leq i \leq k\} \cup \{\{y_1, ..., y_k\}\}$, where $X = \{1, x_2, ..., x_k, y_1, ..., y_k\}$. Then for every $w \in N(v)$ and $w \neq v^c, w^c$ is the unique vertex that is in $N(v^c)$ and adjacent to w. Thus, if $\{ v, w\}$ be an edge of this graph and $v \neq w^c$, then the 4-cycle vww^cv^c is the unique 4-cycle that contains this edge, whereas if $w = v^c$, then any 4-cycle vv^cu^cu, where u is adjacent to v, contains this edge. Let $uwvv^c$ be a 3-path in $FHS(2k,k)$ and $u \neq w^c$, then by a similar way that we have seen in the proof of Theorem 2.4 , we can show that the 6-cycle $uwvv^cw^cu^c$ is the unique 6-cycle that contains this 3-path. It is clear that the girth of this graph is 4. The following figure shows $HS(4,2)$ graph and $FHS(4,2)$ graph.



HS(4,2) graph FHS(4,2) graph FHS(4,2) graph

Fig. 2.

Theorem 3.1. *The automorphism group of folded hyper-star graph $FHS(2k,k)$ is identical to the automorphism group of hyper-star graph $HS(2k,k)$.*

Proof. Let $\Gamma_1 = HS(2k,k)$, $\Gamma_2 = FHS(2k,k)$ and $H = NQ$ be the set which is defined in the proof of Theorem 2.4 . Let $\{v,w\} = e$ be an edge of Γ_2 and $h \in NQ$. If e be an edge of Γ_1, then $h(e)$ is an edge of Γ_2. If e is not an edge of Γ_1, then $w = v^c$. Let $h = nq$, $n \in N$, $q \in Q$, then we have $h(e) = \{h(v), h(v^c)\} = \{nq(v), nq(v^c)\} = \{n(v), n(v^c\}$, now since, $\mid n(v) \cap n(v^c) \mid = \mid v \cap v^c \mid = 0$, then $h(e)$ is an edge of the graph Γ_2. It follows that $H = NQ \leq Aut(\Gamma_2)$. Then $\mid Aut(\Gamma_2) \mid \geq \mid NQ \mid = (2k-1)!2$. Let $G = Aut(\Gamma_2)$. If $\{v,w\}$ be an edge of Γ_2 such that $w \neq v^c$, then we will show that $L_{vw} = 1$. Let $g \in L_{v,w}$. Let u be a vertex of Γ_2 of distance 2 from the vertex v. Then there is a vertex t such that utv is a 2-path in the graph Γ_2. If $t = v^c$, then the 4-cycle, $C : uv^cvu^c$ is the unique 4-cycle that contains the 2-path v^cvu^c. On the other hand, the 4-cycle $g(C) = g(u)v^cvu^c$ also contains this 2-path, hence $g(u) = u$. Suppose that $t \neq v^c, u^c$, then the path $utvw$ is a 3-path in the subgraph $\Gamma_1 = HS(2k,k)$, so there is a unique 6-cycle $C : utvwrs$ in Γ_1 that contains this 3-path. C also is the unique 6-cycle in Γ_2 that contains the 3-path $utvw$. On the other hand, $g(C) = g(u)g(t)g(v)g(w)g(r)g(s) = g(u)tvwrg(s)$, thus $g(C)$ and C are 6-cycles that contains the 3-path $tvwr$, hence $g(C) = C$ and $g_{|V(C)}$, the restriction of g to $V(C)$, is an automorphism of the cycle C that fixes the vertices t, v, w, r, therefore $g(u) = u$. If $u = t^c$, then $C : vtt^cv^c$ is the unique 4-cycle that contains the 2-path v^cvt and $g(C) : vtg(t^c)v^c$ also contains this 2-path, so $g(C) = C$, then $g(u) = u$.

Since the graph Γ_2 is a connected graph, thus we can conclude that $g(u) = u$ for any vertex u of Γ_2, then $L_{v,w} = 1$.

Let v be a vertex of Γ_2, since this graph is a regular graph of degree $k+1$, then from Lemma 2.3, it follows that $\mid G_v \mid \leq (k+1)!k!$. Now, We show that in fact, $\mid G_v \mid \leq (k-1)!k!$. Let v be a vertex of Γ_2, $w \in N(v)$ and $w \neq v^c$. If $g \in L_v$, then g fixes w, so g induces a permutation on $N(w)$. Since, the 4-cycles $C : wvv^cw^c$ and $g(C) = wvv^cg(w^c)$, are identical, then

$g(w^c) = w^c$. Therefore g fixes two elements v and w^c of $N(w)$, hence $L_v/L_{vw} \leq Sym(k+1-2)$, thus $\mid L_v \mid \leq (k-1)!$. Now, let $h \in G_v$, then h induces a permutation on $N(v)$, so $h(v^c) = w$ is in $N(v)$. Let $B = N(v) \cup N(v^c) - \{v, v^c\}$ and $S[B] = T$ be the subgraph induced by B. It is clear that T is isomorphic to $h(T)$, where $h(T)$ is the subgraph induced by the set $D = h(B) = N(v) \cup N(w) - \{v, w\}$. We assert that if $w \neq v^c$, then the subgraph induced by D has not any edge, whereas the subgraph induced by B has k edges. Suppose $x, y \in D$ and $x \in N(v)$ and $y \in N(w)$. We can assume that $v = \{1, x_2, ..., x_k\}$ and $w = \{y_i, x_2, ..., x_k\}$, then $x = \{y_j, x_2, ..., x_k\}$ and $y = \{y_i, x_2, ...x_{l-1}, 1, x_{l+1}, ..., x_k\}$, where $i \neq j$. Now, it is clear that $\{x, y\}$ is not an edge of Γ_2. Hence, $h(v^c) = w = v^c$. Now if $Y = N(v) - \{v^c\}$, then $h_{|Y} \in Sym(Y)$, so $G_v/L_v \leq Sym(k)$, therefore $\mid G_v \mid \leq \mid L_v \mid (k!) \leq (k-1)!(k!)$. Since The graph Γ_2 is a vertex transitive graph, thus;

$$\mid Aut(\Gamma_2) \mid = \mid G \mid = \mid V(\Gamma_2) \mid\mid G_v \mid \leq \frac{2k!}{k!k!}(k!)(k-1)! = (2k-1)!2$$

Now, we have $Aut(FHS(2k, k)) = H = NQ = Aut(HS(2k, k))$.

□

If $k = 2$, then $HS(2k, k)$ is isomorphic to C_6, the cycle on 6 vertices, hence $Aut(HS(4, 2))$ is D_{12}, the dihedral group of order 12. If m be an odd number, then $D_{4m} = D_{2m} \rtimes Z_2$. Therefore $D_{12} = D_6 \rtimes Z_2$, but $D_6 \cong Sym(3)$, hence Theorem 2.4 is also true for $k = 2$. But $FHS(4, 2)$ is isomorphic to $K_{3,3}$, the complete bipartite graph of degree 3, and $Aut(K_{3,3})$ is a group of order 72 [2], thus Theorem 2.5 is not true for $k = 2$.

The group G acting on a set Ω induces a natural action on the set $\Omega^{\{m\}}$, the set of m-element subsets of Ω, by the rule $A^g = \{a_1, ..., a_m\}^g = \{g(a_1), ..., g(a_m)\}$, where $A \subseteq \Omega$ and $g \in G$. The group G is called m-homogenous, if its action on $\Omega^{\{m\}}$ is transitive. We need the following fact.

FACT [9]. Let G be a group acting on a set Ω, and $|\Omega| = n \geq 2m$, $m \geq 2$. If G is m-homogenous, then it is also $(m-1)$-homogenous.

Theorem 3.2. *Let $k \geq 3$. If $\Gamma \in \{HS(2k,k), FHS(2k,k)\}$, then Γ is not a Cayley graph.*

Proof. We know that $Aut(HS(2k,k)) = Aut(FHS(2k,k))$, so if R is a subgroup of $Aut(HS(2k,k))$, then R acts regularly on $V(HS(2k,k))$ if and only if R acts regularly on $V(FHS(2k,k))$. Hence, it is enough to prove the theorem for $HS(2k,k)$. Suppose the contrary, that $HS(2k,k)$ is a Cayley graph, then $Aut(HS(2k,k))$ has a subgroup R that acts regularly on $V(HS(2k,k))$, then $\mid R \mid= \binom{2k}{k} = \frac{2k!}{k!k!}$. If r is an element of R, then $r = \tilde{\sigma}\theta^i$, where $\tilde{\sigma}$ and θ are defined in the proof of Theorem 2.4 and $i \in \{0,1\}$. Let $M_1 = \{\tilde{\sigma} \mid \tilde{\sigma} \in R\}$, then M_1 is a subgroup of R. Since R acts on $V(HS(2k,k))$ transitively, so R contains an element of the form $\tilde{\sigma}\theta$. Now, if $M_2 =\{\tilde{\alpha}\theta | \tilde{\alpha}\theta \in R\}$, then $M_2\tilde{\sigma}\theta \subseteq M_1$, because $\tilde{\alpha}\theta\tilde{\sigma}\theta = \tilde{\alpha}\tilde{\gamma}$. Then, $|M_2| \leq |M_1|$. Since $M_1\tilde{\sigma}\theta \subseteq M_2$, then $|M_1| \leq| M_2|$, so $|M_1| = |M_2| = (1/2)R$. If $M = \{\sigma \mid \tilde{\sigma} \in M_1\}$, then $\mid M_1 \mid=\mid M \mid$ and M is a subgroup of $Sym(X)$ and every element of M fixes the element 1, where $X = \{1, 2, ..., 2k\}$. In fact M acts on $Y = \{2, ..., 2k\}$ and is $(k-1)$-homogenous on this set. Since $2(k-1) \leq 2k-1$, then M is $(k-2)$-homogenous on Y. Hence we must have, $\binom{2k-1}{k-2} \mid \mid M \mid= (1/2)\binom{2k}{k}$, therefore $2(2k-1!)k!k! \mid (2k!)(k-2)!(k+1)!$, hence $k(k-1) \mid k(k+1)$, so $k-1 \mid k+1$, thus we must have $k \in \{1, 2, 3\}$. If $k = 3$, then $\mid M \mid =(1/2)\binom{6}{3} = 10$. Since $2 \mid |M \mid$, then there is an element σ in M such that the order of σ is 2. Note that σ is an element of $Sym(6)$ that fixes 1. If we write σ in the form of a product of disjoint cycles, then $\sigma = (rs)$ or $\sigma = (rs)(tu)$, where $r, s, t, u \in \{2, 3, ..., 6\}$. In each of these cases, for $\tilde{\sigma} \in R$ and vertex $v = \{1, r, s\}$ of $HS(6,3)$ we have $\tilde{\sigma}(v) = \{\sigma(1), \sigma(r), \sigma(s)\} = \{1, r, s\} = v$. Thus R can not be a regular subgroup of $Aut(HS(6,3))$ which contradicts the assumption.

□

If $k = 2$, then $HS(4,2)$ is C_6, the cycle on 6 vertices, which is the Cayley graph $\Gamma = \Gamma(Z_6, \Omega)$, where Z_6 is the cyclic group of order 6 and $\Omega = \{1, -1\}$. The graph $FHS(4,2)$ is $K_{3,3}$, the complete bipartite graph of degree 3, which is the Cayley graph $\Gamma = \Gamma(Sym(3), \Omega)$, where $\Omega = \{(12), (23), (13)\}$ [2].

References

[1] S. B. Akers and B. Krishnamurthy, A group-theoretic model for symmetric interconnection networks, IEEE Trans. Comput. 38 (1989), no. 4, 555-566.

[2] N. L. Biggs, Algebraic Graph Theory (Second edition), Cambridge Mathematical Library (Cambridge University Press, Cambridge, 1993).

[3] E. Cheng and L. liptak, Structural properties of hyper-stars, Ars Combinatoria 80 (2006), pp. 65-73.

[4] S.A. Choudum and V. Sunitha, Automorphisms of augmented cubes, International Journal of Computer Mathematics . 85, No. 11, 2008, 1621-1627.

[5] C. Godsil, G. Royle, Algebraic Graph Theory, Springer (2001).

[6] F. Harary, The Automorphism Group of a Hypercube, Journal of Universal Computer Science. 6, no. 1 (2000), 136-138.

[7] Jong-Seok Kima, Eddie Cheng, Laszlo Liptkb and Hyeong-Ok Lee, Embedding hypercubes, rings, and odd graphs into hyper-stars, International Journal of Computer Mathematics. 86, No. 5, 2009, 771-778.

[8] Jong-Seok Kim, Eunseuk Oh, Hyeong-Seok Lim, and Teong-Nam Heo. Topological and communication aspects of hyper-star graphs. In lecture Notes in Comput. sci. 2869, pages 51-58. Springer, Berlin, 2003.

[9] D. Livingstone, A. Wagner, Transitivity of finite permutation groups on unordered sets. Math. Z. 90, (1965) 393-403.

[10] Rotman, J. J., An Introduction to the Theory of Groups, 4th ed., Springer-Verlag, New York, 1995.

[11] M. Watkins, Connectivity of transitive graphs, J. Combin. Theory 8 (1970) 23-29.

Department of Mathematics, University of Isfahan, Isfahan 81746-73441, Iran

E-mail address: `smortezamirafzal@yahoo.com`

E-mail address: `mirafzal@math.ui.ac.ir`