Balanced *House*-systems and Nestings

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Abstract

A G-design is called *balanced* if the degree of each vertex x is a constant. A G-design is called *strongly balanced* if for every $i = 1, 2, ..., h$, there exists a constant C_i such that $d_{A_i}(x) = C_i$ for every vertex x, where A_is are the orbits of the automorphism group of G on its vertex-set and $d_{A_i}(x)$ of a vertex is the number of blocks containing x as an element of A_i. We say that a G-design is simply balanced if it is balanced, but not strongly balanced. In this paper we determine the spectrum for simply balanced and strongly balanced *House*-systems. Further, we determine the spectrum for House-systems of all admissible indices nesting C_4-systems.

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1 Introduction

Let λK_v be the complete undirected graph defined on the vertex set X, having λ edges for every pair of vertices. Let G be a subgraph of λK_v. A *G-decomposition* of λK_v is a pair $\Sigma = (X, \mathcal{B})$, where $\mathcal{B}$ is a partition of the edge set of λK_v into subsets isomorphic to G. A G-decomposition of λK_v is also called a *G-design* of *order* v, *index* λ, and the classes of the partition $\mathcal{B}$ are said to be the *blocks* of Σ [10, 11, 12].

A G-design is called *balanced* if for each vertex $x \in X$, the number of blocks of Σ containing x is a constant. Observe that if G is a regular graph then a G-design is always balanced, hence the notion of a balanced G-design becomes meaningful only for a non-regular graph G.

Let G be a graph and let $A_1, A_2, ..., A_h$ be the orbits of the automorphism group of G on its vertex-set. Let $\Sigma = (V, \mathcal{B})$ be a G-design. We define the degree $d_{A_i}(x)$ of a vertex $x \in X$ as the number of blocks of Σ containing x as an element of A_i. We say that $\Sigma = (X, \mathcal{B})$ is a *strongly balanced* G-design if, for every $i = 1, 2, ..., h$, there exists a constant C_i such that $d_{A_i}(x) = C_i$, for every $x \in X$.

Clearly, since for each vertex $x \in X$ the relation $d(x) = \sum_{i=1}^{h} d_{A_i}(x)$ holds, it follows that *a strongly balanced G-design is a balanced G-design.* We say that a G-design is *simply balanced* if it is balanced, but not strongly balanced [5, 7, 8].

A cycle of length 5 with a *chordal* (edge joining two not adiacent vertices) will said to be an *house-graph*, denoted by H_5. If $H_5 = (V, E)$, with $V = \{a, b, c, d, e\}$ and $E = \{\{a, b\}, \{b, c\}, \{c, d\}, \{d, e\}, \{a, e\}, \{a, c\}\}$, we will denoted such a graph by $[(a), b, (c), d, e]$ or also $[b, (c), d, e, (a)]$, $[(c), d, e, (a), b]$ or similar. We can see that H_5 admits three orbits: $A_1 = \{b\}$, $A_2 = \{a, c\}$, $A_3 = \{d, e\}$.

An H_5-design or H_5-system $\Sigma = (X, \mathcal{B})$ of order v and index λ is said to be C_4*-perfect*, if the family of all the C_4-cycles having, for every block H_5, edges $\{a, c\}, \{c, d\}, \{d, e\}, \{e, a\}$ generates a C_4-design $\Sigma' = (X, \mathcal{B}')$ of order v and index μ. In these cases, Σ' is said to be *nested* in Σ and also that Σ is *nesting* Σ'. Further, we say that Σ has indices (λ, μ).

In this paper we will determine the spectrum for strongly balanced H_5-designs, for simply balanced H_5-designs, and the spectrum for C_4*-perfect* H_5-designs.

Some balanced G-designs when G is a path are studied in [1, 5, 9]. Strongly balanced G-designs were first introduced in [3], in which the spectrum of simply balanced and strongly balanced P_5 and P_6-designs were determined, where P_k denotes a path with k vertices. The spectrum of simple and strongly balanced P_k-designs has been determine in [5] and $4 - kite$-designs has been determined in [7]. For nested G'-designs see [2, 3, 4, 6, 9, 11].

2 Balanced H_5-designs

In this section we determine the spectrum for strongly balanced H_5-designs and simply balanced H_5-designs. We will consider the case of H_5-designs having index $\lambda = 1$. Observe that it is well known the spectrum of H_5-designs. Indeed, *an* H_5*-design of order* v *exists if and only if* $v \equiv 0,1,4$ *or* 9 (*mod* 12).

The following examples give a strongly balanced H_5-design, a simply balanced H_5-design and a not balanced H_5-design, all of order $v = 13$.

Example 2.1. *Strongly balanced* H_5*-design of order* 13.

Let $\Sigma = (X, \mathcal{B})$ be the H_5-design defined in Z_{13}, where $\mathcal{B} = \{[(i), i+4, (i+1), i+7, i+2] \mid i \in Z_{13}\}$.
We can verify that Σ is an H_5-design of order 13. Further, Σ is strongly balanced. Indeed, for every vertex $x \in X$ it is: $d_{A_1}(x) = 1$, $d_{A_2}(x) = 2$, $d_{A_3}(x) = 2$.

Example 2.2. *Simply balanced* H_5*-design of order* 13.

Let $\Sigma = (X, \mathcal{B})$ be the H_5-design defined in Z_{13}, where

$$\mathcal{B} = \{[(i+2), i+1, (i+4), i+11, i+6] \mid i = 0, 1, ..., 9\} \cup \\ \{[(0), 12, (2), 9, 4], [(1), 8, (3), 10, 5], [(1), 11, (12), 3, 0]\}.$$

We can verify that Σ is an H_5-design of order 13. Further, since every vertex of Σ has degree 5, Σ is balanced. However, Σ is not strongly balanced. Indeed:

$$d_{A_1}(0) = 0, d_{A_2}(0) = 2, d_{A_3}(0) = 3,$$

$$d_{A_1}(1) = 1, d_{A_2}(1) = 2, d_{A_3}(1) = 2.$$

Example 2.3. *Not balanced* H_5*-design of order* 13.

Let $\Sigma = (X, \mathcal{B})$ be the H_5-design defined in Z_{13}, having the following blocks:

$[(0), 1, (2), 10, 3]$, $[(0), 4, (5), 7, 6]$, $[(0), 7, (8), 4, 9]$,

$[(0), 10, (11), 1, 12]$, $[(1), 3, (4), 6, 5]$, $[(1), 8, (6), 9, 7]$,

$[(9), 2, (5), 10, 1]$, $[(4), 2, (7), 12, 10]$, $[(11), 4, (12), 3, 2]$,

$[(7), 3, (11), 6, 10]$, $[(11), 8, (9), 3, 5]$, $[(2), 12, (6), 3, 8]$,

$[(8), 5, (12), 9, 10]$.

Σ is an H_5-design of order 13. We can see that $d(0) = 4$ and $d(3) = 6$. Therefore it is not balanced.

Theorem 2.4. *The necessary condition for the the existence of a balanced H_5-design of order v is $v \equiv 1$ $(mod\ 12)$, $v \geq 13$.*

Proof - Let $\Sigma = (X, \mathcal{B})$ be a balanced H_5-design of order v. This means that all the vertices of Σ have the same degree C. Considering that: $5 \cdot |\mathcal{B}| = C \cdot v$, because of every block of Σ contains five vertices, and that: $|\mathcal{B}| = v(v-1)/12$, because of $v(v-1)/2$ is the total number of pairs of elements of X and in every block there are six of these pairs, it follows:

$$C = \tfrac{5(v-1)}{12},$$

from which: $v \equiv 1$ (mod12), $v \geq 13$, necessarily. □

In what follows, to simplify, for every vertex x of an H_5-design of order v, we will indicate by U_x *(upper)*, M_x *(middle)*, L_x *(lower)*, the degrees:

$$U_x = d_{A_1}(x), M_x = d_{A_2}(x), L_x = d_{A_3}(x).$$

Theorem 2.5. *If $\Sigma = (X, \mathcal{B})$ is a a strongly balanced H_5-design of order v, then:*
$\forall x \in X, \quad U_x = \frac{v-1}{12}, M_x = L_x = \frac{v-1}{6}.$

Proof - Since each vertex x has degree $d_{A_1}(x) = U$, it follows: $U \cdot v = \frac{v(v-1)}{12}$, from which: $U = \frac{v-1}{12}$. Similarly, each vertex has also degrees $d_{A_2}(x) = M$ and $d_{A_3}(x) = L$. Therefore: $M \cdot v = L \cdot v = 2 \cdot \frac{v(v-1)}{12}$, from which: $M = L = \frac{v-1}{6}$. □

Theorem 2.6. *The necessary condition for the the existence of a strongly balanced H_5-design, having order v, is $v \equiv 1$, $(mod\ 12)$, $v \geq 13$.*

Proof - From conditions of Theorem 2.5. □

The previous Theorems are completed by the following characterizations.

Theorem 2.7. *There exists a strongly balanced H_5-design if and only if $v \equiv 1$ $(mod\ 12)$, $v \geq 13$.*

Proof - The necessity follows from Theorem 2.6. For the sufficiency, let $\Sigma = (X, \mathcal{B})$ be the H_5-design defined in $\mathbb{Z}_{12k+1}$, where the blocks are obtained by the following k base blocks in modulo $12k + 1$, $k \geq 1$:

$$[(2i+1), 0, (4k-2i-1), 8k+2i+2, 4k-2i+1],$$

for every $k \geq 1$ and $i = 0, 1, 2, ..., k-1$.

Since each difference 1, 2, 3,..., $6k$ is covered exactly once, Σ is an H_5-design of order $v = 12k+1$. Further, it is possible to verify that for every vertex $x \in X$ we have:

$U_x = k$, $M_x = 2k$, $L_x = 2k$.

Therefore, Σ is a strongly balanced system. □

Theorem 2.8. *There exists a simply balanced H_5-design if and only if $v \equiv 1$ (mod 12), $v \geq 13$.*

Proof - For $v = 13$, the H_5-design considered in Example 2.2 is balanced, but not strongly balanced.

Let $v = 12k+1$, $k \geq 2$, and define $\Gamma = (\mathbb{Z}_{12k+1}, \mathcal{C})$ having for blocks all the translates of the following base blocks:

$$[(2i+1), 0, (4k-2i-1), 8k+2i+2, 4k-2i+1], \text{ for } i = 0, \ldots, k-2;$$

$$[(2k-1), 0, (2k+1), 10k, 2k+3].$$

We can verify that Γ is an H_5-design of order $v = 12k+1$. Further, because of used construction, Γ is strongly balanced.

Now, consider the following two block of $\mathcal{C}$:

$$C_1 = [(2k-1), 0, (2k+1), 10k, 2k+3],$$

$$C_2 = [(2k-3), 12k-1, (2k-1), 10k-2, 2k+1].$$

If

$$B_1 = [(2k-1), 10k-2, (2k+1), 10k, 2k+3],$$

$$B_2 = [(2k-3), 12k-1, (2k-1), 0, 2k+1],$$

and $\mathcal{B} = \mathcal{C} - \{C_1, C_2\} \cup \{B_1, B_2\}$, we can verify that also $\Sigma = (\mathbb{Z}_{12k+1}, \mathcal{B})$ is an H_5-design of order $v = 12k+1$.

Further, in Σ it happens that every vertex $x \in \mathbb{Z}_{12k+1}$ has degree: $d(x) = 5k = \frac{5(v-1)}{12}$, therefore Σ is balanced.

To prove that Σ is not strongly balanced, it is sufficient to examine the vertices $x = 0$ and $y = 10k-2$. We have:

$U(0) = k-1, M(0) = 2k, L(0) = 2k+1;$

$U(10k-2) = k+1, M(10k-2) = 2k, L(10k-2) = 2k-1$.
Therefore, Σ is a simply balanced system. □

3 C_4-perfect H_5-designs

We begin this section determining some necessary existence conditions for C_4-perfect H_5-designs, briefly C_4-perfect-H_5-D. In what follows, if $\Omega = (X, \mathcal{B})$ is a C_4-perfect H_5-design of order v and index λ, nesting a C_4-design $\Sigma = (X, \mathcal{D})$ of index μ, we will say that Ω has indices (λ, μ).

Theorem 3.1. *If $\Omega = (X, \mathcal{B})$ is a C_4-perfect H_5-design of order v, nesting a C_4-design $\Sigma = (X, \mathcal{D})$, having indices (λ, μ), then: $2\lambda = 3\mu$.*

Proof - Since $|\mathcal{B}| = \frac{v(v-1)}{12} \cdot \lambda$, $|\mathcal{D}| \frac{v(v-1)}{8} \cdot \mu$, and necessarily $|\mathcal{B}| = |\mathcal{D}|$, it follows:

$$\frac{v(v-1)}{12} \cdot \lambda = \frac{v(v-1)}{8} \cdot \mu,$$

Hence: $2\lambda = 3\mu$. □

We begin to consider the first case: $\lambda = 3, \mu = 2$.

Theorem 3.2. *Let $\lambda = 3$, $\mu = 2$. There exists a C_4-perfect-H_5-D of order v and indices $(3, 2)$ if and only if $v \equiv 0, 1 \mod 4$, $v \geq 5$.*

Proof. Let $v = 4k+1$, $k \geq 1$. Let us consider the following k *base* blocks:

$$[B_i = [(0), 2i+2, (4i+3), 8i+4, 4i+1] \qquad \text{for } i = 0, \ldots, k-1.]$$

If $\Sigma = (\mathbb{Z}_{4k+1}, \mathcal{B})$ is the system having for blocks all the translates of the blocks B_i, then we can verify that Σ is a C_4-perfect-H_5-D of order $v = 4k+1$.

Let $v = 4k$, $k \geq 2$. Let us consider $\Sigma = (\mathbb{Z}_{4k-1} \cup \{\infty\}, \mathcal{B})$, where $\infty \notin \mathbb{Z}_{4k-1}$ and $\mathcal{B}$ has for blocks all the translates of the following base blocks in which ∞ is a *fixed* point:

$C = [(\infty), 0, (1), 3, 2]$;

$D_i = [(2i), 0, (4k-2i-2), 2i+1, 4k-2i]$, for any $i \in \{1, \ldots, k-1\}$.

We can verify that Σ is a C_4-perfect-H_5-D of order $v = 4k$. □

Now, we consider the case: $\lambda = 6, \mu = 4$.

Theorem 3.3. *Let $\lambda = 6$ and $\mu = 4$. There exists a C_4-perfect-H_5-D of order v and indices $(6,4)$ for every integer $v \geq 5$.*

Proof. If $v \equiv 0, 1 \pmod 4$ the C_4-perfect-H_5-D of order v can be obtained by the systems constructed in Theorem 3.2 by a repetition of blocks.

If $v = 4k+3$, $k \geq 1$, consider the system $\Sigma = (\mathbb{Z}_{4k+3}, \mathcal{B})$, having for blocks all the translates of the following base blocks:

$B_i = [(i), 0, (2i), i-1, 4k+2]$, for any $i = 2, \ldots, 2k$;

$C = [(1), 0, (2), 4, 3]$;

$D = [(1), 0, (2k+2), 2k+3, 2]$.

We can verify that Σ is a C_4-perfect-H_5-D of order v and indices $(6,4)$.

If $v = 4k+2$, $k \geq 1$, consider the system $\Sigma = (\mathbb{Z}_{4k+1} \cup \{\infty\}, \mathcal{B})$, with $\infty \notin \mathbb{Z}_{4k+1}$, having for blocks all the blocks of the C_4-perfect-H_5-D of indices $(3,2)$ and order $v = 4k+1$ given in Theorem 3.2 and the translates of the following base blocks, where ∞ is a fixed point:

$A = [(\infty), 0, (1), 3, 2]$;

$B = [(\infty), 0, (2k+1), 2k+3, 2k+2]$;

if $k \geq 2$: $C = [(k), 0, (3k), k+1, 3k+2]$;

if $k \geq 3$: $D_i = [(i), 0, (2k+1-i), 4k-3i+3, 2k-i+2]$, for any $i \in \{2, \ldots, k-1\}$.

We can verify that Σ is a C_4-perfect-H_5-D of order $v = 4k+2$ and indices $(6,4)$. $\square$

In conclusion, from the previous Theorem it follows that:

Theorem 3.4. *There exists a C_4-perfect-H_5-D of order v, $v \geq 5$, and indices (λ, μ), for every pair of indices (λ, μ) such that $2\lambda = 3\mu$.*

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