On pairs of graphs having $n-2$ cards in common

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Abstract

For a vertex v of a graph G, the unlabeled subgraph $G-v$ is called a *card* of G. We prove that the number of isolated vertices and the number of components of an n vertex graph G can be determined from any collection of $n-2$ of its cards for $n \geq 10$. It is also proved that if two graphs of order $n \geq 6$ have $n-2$ cards in common, then the number of edges in them differ by at most one.

Key words: vertex-deleted subgraph (card), common cards, reconstruction.

1 Introduction

All graphs considered are finite, simple and undirected. We use the terminology in Harary [2]. The degree of a vertex v and the minimum degree among the vertices of a graph G are denoted by $deg_G(v)$ (or simply $deg\ v$) and $\delta(G)$ (or simply δ), respectively. The number of edges of G is denoted by $e(G)$. The set of all neighbours of a vertex v of G is denoted by $N(v)$. By $u \sim v$, we mean the vertices u and v are adjacent; otherwise we denote $u \not\sim v$.

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For a vertex v of G, the unlabeled subgraph $G-v$ is called a *card* of G. A graph G on $n \geq 3$ vertices is *reconstructible* if there is no graph H not isomorphic to G with n cards in common with G. The famous Reconstruction Conjecture claims that all graphs on three or more vertices are reconstructible. Manvel [3] proved that, if G and H are graphs on n vertices with $n-1$ cards in common, then $|e(G)-e(H)| \leq 1$. Myrvold [5] showed that for $n \geq 7$, $e(G)=e(H)$ whenever G and H have $n-1$ cards in common. For $3 \leq n \leq 5$, there are graph pairs G and H on n vertices with $n-2$ cards in common such that $|e(G)-e(H)|=2$. But for $n \geq 6$, it was proved [6] that if two graphs G and H on n vertices and $\delta \geq 2$ have $n-2$ cards in common, then $|e(G)-e(H)| \leq 1$.

In this paper, we prove that the number of isolated vertices and the number of components of G are reconstructible from any collection of $n-2$ of its cards for $n \geq 10$. It is also proved that, if G and H are graphs on $n \geq 6$ vertices with $n-2$ cards in common, then $|e(G)-e(H)| \leq 1$.

2 Reconstruction from $n-2$ cards

Here we prove that the number of isolated vertices of G (denoted by $n_0(G)$) and the number of components of G (denoted by $\omega(G)$) are reconstructible from any collection of $n-2$ of its cards.
We first recall the following two lemmas.

Lemma 1 ([6]). Connected graphs and disconnected graphs on n (≥ 7) vertices are recognizable from any collection of $n-2$ of their cards.

Lemma 2 ([6]). A graph with an isolated vertex can be recognized from any collection of $n-2$ of its cards for $n \geq 7$.

Theorem 3. The number of isolated vertices in a graph G of order n is reconstructible from any collection of $n-2$ of its cards for $n \geq 10$.

Proof. In view of Lemmas 1 and 2, we can take that G is disconnected with isolated vertices. Let $\mathscr{F}$ be the given collection of $n-2$ cards of G.

If either $\mathscr{F}$ contains a card with no isolated vertices, or at least three cards in $\mathscr{F}$ have exactly one isolated vertex, then $n_0(G)=1$ (since a graph with at least two isolated vertices have exactly two cards with exactly one isolated vertex). We assume, therefore, that every card in $\mathscr{F}$ has isolated vertex, and that at most two cards in $\mathscr{F}$ have exactly one isolated vertex.

Case 1. There is a card in $\mathscr{F}$ with exactly one isolated vertex.

Then $n_0(G) = 1$ or 2. Let us characterize G when it has exactly one isolated vertex.

Since all but at most three cards of G have at least two isolated vertices, G must have at least $n-3$ vertices adjacent to a vertex of degree one and hence, *at least* $n-3$ *vertices of degree one.* Also G can have *at most one component on three or more vertices* (since each such component of G gives rise to at least 2 cards of G with exactly one isolated vertex). Among the other components of G, one is K_1 and the remaining are K_2 's.

If G *has no components on three or more vertices*, then it is $(\frac{n-1}{2})K_2 \cup K_1$, leading to a contradiction (since $\mathscr{F}$ contains a card with exactly one isolated vertex).

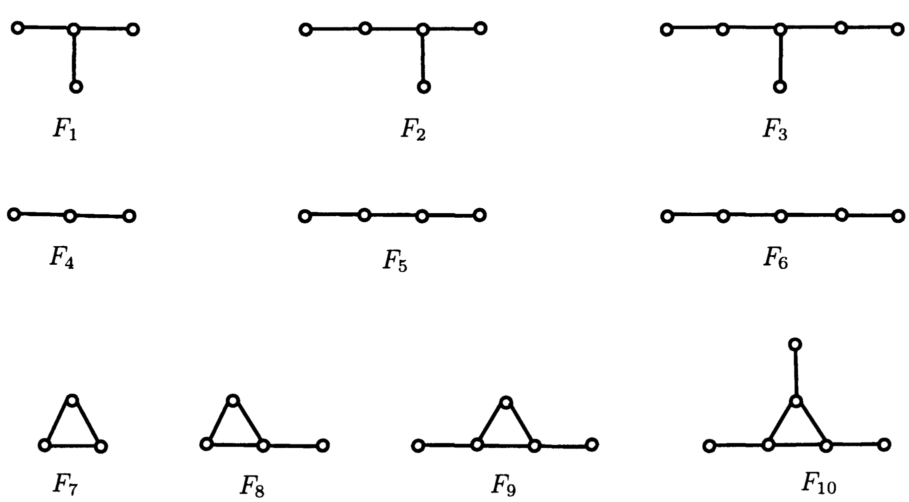

Figure 1: Possibilities for the sole non K_2 component of G

If G *has a component which is a tree, and that tree has four or more endvertices*, then there would be at least four cards with exactly one isolated vertex (which is not the case here). So any components which are trees have at most three endvertices. If *the tree has three endvertices*, then it has one vertex v of degree three. If v is not adjacent to some endvertex, then the graph again has at least four cards with exactly one isolated vertex (those obtained from deleting v and the three endvertices). Thus the degree three vertex v is adjacent to an endvertex. If there is a path of length three or more from v to one of the other endvertices which has first edge vu, the four cards obtained from deleting the three endvertices or u have exactly one isolated vertex. Thus, if G has a component which is a tree having three endvertices, this component must be one of F_1, F_2

and F_3 of Figure 1.

If *G has a component which is a tree having only two endvertices* (the only remaining case for trees) then it is a path. Any path on six or more vertices has at least four cards which have exactly one isolated vertex. So such a component is one of F_4, F_5 and F_6 of Figure 1.

The remaining cases are when G has a component on three or more vertices which has a cycle. Such a cycle can have size at most three. The reason is that either a cycle vertex v is adjacent to an endvertex or it is not. If the vertex v is adjacent to an endvertex u, then $G-u$ has exactly one isolated vertex. If it is not, then $G-v$ has exactly one isolated vertex. Similar reasoning can be used to argue that the component only has one cycle. Let the cycle be (a,b,c). These vertices may or may not be adjacent to an endvertex. But either way, there can be no other vertices besides $a,b,c,$ and at most one endvertex adjacent to each which are in this component or else there will be four or more cards with exactly one isolated vertex. The possible cases are F_7, F_8, F_9 or F_{10} of Figure 1.

Thus *when G has exactly one isolated vertex*, it has to be one among the ten graphs $F_1\cup(\frac{n-5}{2})K_2\cup K_1$, $F_2\cup(\frac{n-6}{2})K_2\cup K_1$, $F_3\cup(\frac{n-7}{2})K_2\cup K_1$, $F_4\cup(\frac{n-4}{2})K_2\cup K_1$, $F_5\cup(\frac{n-5}{2})K_2\cup K_1$, $F_6\cup(\frac{n-6}{2})K_2\cup K_1$, $F_7\cup(\frac{n-4}{2})K_2\cup K_1$, $F_8\cup(\frac{n-5}{2})K_2\cup K_1$, $F_9\cup(\frac{n-6}{2})K_2\cup K_1$, and $F_{10}\cup(\frac{n-7}{2})K_2\cup K_1$. Hence if $\mathscr{F}$ coincides with some collection of $n-2$ cards of one of these ten graphs, then $n_0(G)=1$. Otherwise, $n_0(G)=2$.

Case 2. Each card in $\mathscr{F}$ has at least two isolated vertices.

If at least four cards in $\mathscr{F}$ have exactly two isolated vertices, then $n_0(G)=2$. Hence we can take that *at most three cards in $\mathscr{F}$ have exactly two isolated vertices.* Then $n_0(G)\geq 2$ (as otherwise G has exactly one isolated vertex (say x). Since $\mathscr{F}$ contains a card with at least three isolated vertices, G has a component (say C) on three or more vertices. Then at least three cards of G, obtained from deleting x or the non-cut vertices of C, have at most one isolated vertex, contradicting Case 2).

Subcase 2.1. If there exists a card in $\mathscr{F}$ with exactly two isolated vertices.

Then $n_0(G)=2$ or 3. It can be proved similarly (as in Case 1) that *when G has exactly two isolated vertices*, it has to be one among the ten graphs $F_1\cup(\frac{n-6}{2})K_2\cup 2K_1$, $F_2\cup(\frac{n-7}{2})K_2\cup 2K_1$, $F_3\cup(\frac{n-8}{2})K_2\cup 2K_1$, $F_4\cup(\frac{n-5}{2})K_2\cup 2K_1$, $F_5\cup(\frac{n-6}{2})K_2\cup 2K_1$, $F_6\cup(\frac{n-7}{2})K_2\cup 2K_1$, $F_7\cup(\frac{n-5}{2})K_2\cup 2K_1$, $F_8\cup(\frac{n-6}{2})K_2\cup 2K_1$, $F_9\cup(\frac{n-7}{2})K_2\cup 2K_1$, and $F_{10}\cup(\frac{n-8}{2})K_2\cup 2K_1$. Hence, if $\mathscr{F}$ coincides with some collection of $n-2$ cards of one these ten graphs, then $n_0(G)=2$. Otherwise, $n_0(G)=3$.

Subcase 2.2. Each card in $\mathscr{F}$ has at least three isolated vertices.

If *there exists a card in* $\mathscr{F}$ *with at least four isolated vertices*, then $n_0(G) \geq 3$ (as otherwise G has exactly two isolated vertices (say x, y). Since $\mathscr{F}$ contains a card with at least four isolated vertices, G has a component (say C) on three or more vertices. Then at least four cards of G, obtained from deleting x or y or the non-cut vertices of C, have at most two isolated vertices, contradicting Subcase 2.2). Therefore at least one card of G obtained from deleting an isolated vertex must belong to $\mathscr{F}$ and hence $n_0(G) = \min_{F \in \mathscr{F}} \{n_0(F)\} + 1$. We assume, therefore, that *each card in* $\mathscr{F}$ *has exactly three isolated vertices.* Then $n_0(G) = 2$ or 3.

Let us characterize G when it has exactly two isolated vertices. Since all but at most two cards of G have exactly three isolated vertices, it follows that G must have at least $n-2$ endvertices adjacent to distinct vertices, and hence $G \cong (\frac{n-2}{2})K_2 \cup 2K_1$. Thus, if $\mathscr{F}$ coincides with some collection of $n-2$ cards of $(\frac{n-2}{2})K_2 \cup 2K_1$, then $n_0(G) = 2$. Otherwise, $n_0(G) = 3$. ∎

Theorem 4. The number of components of G of order $n (\geq 10)$ is reconstructible from any collection of $n-2$ of its cards.

Proof. In view of Lemma 1, we can take that G is disconnected. Let $\mathscr{C}$ be the given collection of $n-2$ cards of G. We consider four cases depending on $n_0(G)$ (this value can be determined from $\mathscr{C}$ (Theorem 3)).

Case 1. $n_0(G) > 2$.

Then $\mathscr{C}$ must contain a card obtained by deleting an isolated vertex of G, and such a card can be identified in $\mathscr{C}$ as a card $G-v$ with $n_0(G-v) = n_0(G) - 1$. Hence $\omega(G) = \omega(G-v) + 1$.

Case 2. $n_0(G) = 2$.

If $\mathscr{C}$ contains a card, say F with exactly one isolated vertex, then F must be obtained from deleting an isolated vertex of G, which implies $\omega(G) = \omega(F) + 1$. Hence, we can take that *each card in* $\mathscr{C}$ *has at least two isolated vertices.* Consequently, $\mathscr{C}$ *does not contain the two cards obtained by deleting the isolated vertices of* G. Then, the collection of cards obtained by deleting exactly two isolated vertices from every card in $\mathscr{C}$ must be the full deck of the graph $H = G - 2K_1$. But it is well known that the number of components can be determined from the full deck of a graph. Thus $\omega(H)$ and hence $\omega(G)$ can be determined.

Case 3. $n_0(G) = 1$.

We can take that each card in $\mathscr{C}$ has at least one isolated vertex (as otherwise $\omega(G) = \min_{F\in\mathscr{C}}\{\omega(F)\} + 1$). Then, $\mathscr{C}$ *does not contain the card obtained by deleting the unique isolated vertex of* G. So, the collection $\mathscr{T}$ of $n-2$ cards obtained by deleting exactly one isolated vertex from every card in $\mathscr{C}$ is nothing but the collection of all but one card of the graph $H = G - K_1$. Since every component of H is nontrivial and contains at least two non-cutvertices of H, it follows that $\mathscr{T}$ must contain a card obtained by deleting a non-cutvertex of H. Thus, $\omega(H) = \min_{E\in\mathscr{T}}\{\omega(E)\}$ and $\omega(G) = \omega(H) + 1$.

Case 4. $n_0(G) = 0$.

Since G is disconnected and containing no isolated vertices, it contains at least four non-cutvertices and hence $\omega(G) = \min_{F\in\mathscr{C}}\{\omega(F)\}$. ∎

3 Graphs with $n-2$ cards in common

Whenever G and H are taken as two graphs having $n-2$ cards in common, we assume that G and H are labeled with $v_1, v_2, ..., v_n$ and $u_1, u_2, ..., u_n$ respectively so that $G - v_i \cong H - u_i$ for $i = 1$ to $n-2$. A card $G - v_i$, $1 \le i \le n-2$ is called a *common card of* G and a card $H - u_i$, $1 \le i \le n-2$ is called a *common card of* H.

We recall the following two lemmas. Since the proof of Lemma 5 is very short, we include it.

Lemma 5 ([5]). Let G and H be graphs with $e(G)$ and $e(G)+k$, $k \ge 0$ edges respectively. If $G - v \cong H - u$, then $deg\ u = deg\ v + k$.

Proof. Since $G - v \cong H - u$, it follows that $e(G-v) = e(H-u)$, which implies $e(G) - deg\ v = e(G) + k - deg\ u$ and hence $deg\ u = deg\ v + k$. ∎

Lemma 6 ([6]). Let G and H be graphs on n vertices with $e(G)$ and $e(G)+k$, $k \ge 0$ edges respectively. If G and H have $n-2$ cards in common, then $deg\ v_{n-1} + deg\ v_n - (deg\ u_{n-1} + deg\ u_n) = k(n-4)$. ∎

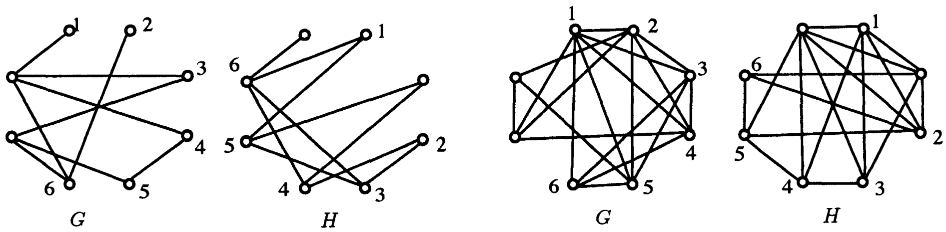

Figure 2: Two Pairs of graphs on eight vertices with six common cards.

Theorem 7. If G and H are graphs on $n \geq 6$ vertices such that they have $n-2$ cards in common, then $|e(G) - e(H)| \leq 1$.

Proof. Rivshin and Radziszowski [7] reported that

(i) for $n = 6, 7$, there exist pairs of nonisomorphic graphs G and H with $n-2$ cards in common (but for which $|e(G) - e(H)| \leq 1$),

(ii) for $n = 8$, there exist only two pairs of nonisomorphic graphs G and H with six common cards (Figure 2) (but for which $|e(G) - e(H)| = 1$), and

(iii) for $n = 9, 10, 11$, there exists no pairs of nonisomorphic graphs with $n-2$ common cards.

Let G and H be two graphs on $n \geq 12$ vertices with $n-2$ common cards. Without loss of generality, let us take $e(G) \leq e(H)$. Since $0 \leq deg_G\ v_i,\ deg_H\ u_i \leq n-1$, it follows that $deg_G\ v_i - deg_H\ u_j \leq n-1$. Therefore

$$deg_G\ v_{n-1} + deg_G\ v_n - (deg_H\ u_{n-1} + deg_H\ u_n) \leq 2(n-1). \quad \ldots\ldots (1)$$

If possible, suppose $e(H) - e(G) \geq 3$. Then by Lemma 6, we have $deg_G\ v_{n-1} + deg_G\ v_n - (deg_H\ u_{n-1} + deg_H\ u_n) \geq 3(n-4)$, which implies from (1) that
$2(n-1) \geq 3(n-4)$. This is impossible as $n \geq 12$.

If possible, let $e(H) - e(G) = 2$. Then by Lemma 5, we have $deg_H\ u_i = deg_G\ v_i + 2$ and hence $deg_H\ u_i \geq 2$ for $i = 1$ to $n-2$. ...... (2)
Now by Lemma 6,
$deg_G\ v_{n-1} + deg_G\ v_n - (deg_H\ u_{n-1} + deg_H\ u_n) = 2n - 8$. ...... (3)
Since $0 \leq deg_G\ v_i \leq n-1$ and $0 \leq deg_H\ u_i \leq n-1$ for $i = 1$ to n, it follows that $deg_G\ v_{n-1} + deg_G\ v_n \leq 2n-2$ and $deg_H\ u_{n-1} + deg_H\ u_n \geq 0$ and (3) gives rise to the seven cases discussed below, each leading to a contradiction. We assume, without loss of generality, that $deg_G\ v_n \geq deg_G\ v_{n-1}$, and $deg_H\ u_n \geq deg_H\ u_{n-1}$.

Case 1. $deg_G\ v_{n-1} + deg_G\ v_n = 2n-2$ and $deg_H\ u_{n-1} + deg_H\ u_n = 6$.

Then $deg_G\ v_{n-1} = deg_G\ v_n = n-1$ and $\{deg_H\ u_{n-1}, deg_H\ u_n\} = \{0,6\}, \{1,5\}, \{2,4\}$, or $\{3\}$. If $\{deg_H\ u_{n-1}, deg_H\ u_n\} \neq \{3\}$, then there must be a common card of H containing a vertex of degree 0 or 1. But no common card of G contains such a vertex (since $deg_G\ v_{n-1} = deg_G\ v_n = n-1$), giving a contradiction. So, we can take that $deg_H\ u_{n-1} = deg_H\ u_n = 3$, and that $|N(u_{n-1}) \cap N(u_n)| = 3$ (as otherwise, there must be at least one common card of H containing at most one $(n-2)$-vertex. But, since $deg_G\ v_{n-1} = deg_G\ v_n = n-1$, it follows that each common card of G has at least two $(n-2)$-vertices, producing a contradiction). Let $N(u_{n-1}) \cap N(u_n) = \{u_r, u_s, u_t\}$.

Now each common card $G - v_i$ $(1 \leq i \leq n-2)$ has at least two adjacent $(n-2)$-vertices, namely v_{n-1}, v_n. Consequently, each $H - u_i$ $(1 \leq i \leq n-2)$ has at least two adjacent $(n-2)$-vertices. But the only vertices that can have degree at least $n-2$ in H are u_r, u_s and u_t. As a result, $deg_H\ u_i = n-1$, for $i \in \{r,s,t\}$. Hence each of the $n-5$ (≥ 6) common cards $H - u_i$ (and hence $G - v_i$), $i \notin \{r,s,t,n-1,n\}$ has three $(n-2)$-vertices. As a result, G has a vertex, say v_k ($k \neq n-1$, n) of degree at least $n-2$. Let v_q be the non-neighbour of v_k. Then $v_q \neq v_{n-1}$, v_n and hence each of the $n-3$ common cards $G - v_i$ ($1 \leq i \leq n-2$ and $i \neq k$) has at most one 2-vertex. Thus, at most one common card of G can have at least two vertices of degree two. This is impossible (since each of the three common cards $H - u_r$, $H - u_s$ and $H - u_t$ have at least two vertices of degree two).

Case 2. $deg_G\ v_{n-1} + deg_H\ v_n = 2n-3$ and $deg_H\ u_{n-1} + deg_H\ u_n = 5$.

Now $deg_G\ v_n = n-1$ and $deg_G\ v_{n-1} = n-2$. Hence in G, there can be at most one vertex which is not adjacent to v_{n-1} and this alone can be endvertex of G. Thus, *G has at most one endvertex and no isolated vertex.*

Since $deg_H\ u_{n-1} + deg_H\ u_n = 5$, it follows that $\{deg_H\ u_{n-1}, deg_H\ u_n\} = \{0,5\}, \{1,4\}$, or $\{2,3\}$. If $deg_H\ u_{n-1} = 0$ or 1, then there must be a common card of H containing a 0-vertex. But no common card of G contains such a vertex (since $deg_G\ v_{n-1} = n-1$), giving a contradiction. We assume, therefore, that $\{deg_H\ u_{n-1}, deg_H\ u_n\} = \{2,3\}$, and that $|N(u_n) \cap N(u_{n-1})| = 2$ (as otherwise, there must be at least one common card of H containing no $(n-2)$-vertex. But every common card of G has at least one $(n-2)$-vertex, again contradicting).

Let $u_a \in N(u_n) \setminus N(u_{n-1})$ and $u_b, u_c \in N(u_n) \cap N(u_{n-1})$; let v_s be the nonneighbour of v_{n-1} in G. Then each of the $n-3$ common cards $G - v_i$ (and hence $H - u_i$) ($1 \leq i \leq n-2$ and $i \neq s$) has an

$(n-2)$-vertex and an $(n-3)$-vertex. The remaining card $G-v_s$ (and hence $H-u_s$) has two adjacent $(n-2)$-vertices. As a result, H must have a vertex of degree at least $n-2$. However vertices in H that can have degree at least $n-2$ are u_a, u_b, and u_c only. If u_b and u_c have degree at most $n-2$ in H, then there exists a common card of H with no $(n-2)$-vertex, contradicting. If one of u_b and u_c, say u_b has degree $n-1$ and u_c has degree at most $n-2$, then the common card $H-u_a$ has no $(n-2)$-vertex, again contradicting. We take, therefore, that $deg_H\ u_b = deg_H\ u_c = n-1$. Since $u_a \sim u_{n-1}$, $deg_H\ u_a \leq n-2$. If $deg_H\ u_a < n-2$, then the common cards $H-u_b$ and $H-u_c$ have no $(n-3)$-vertex. Consequently, at most $n-5$ common cards of H have $(n-3)$-vertex, giving a contradiction. Hence, let $deg_H\ u_a = n-2$. Then exactly two common cards of H (namely $H-u_b$ and $H-u_c$) have an endvertex adjacent to an $(n-2)$-vertex. But at most one or at least $n-3$ common cards of G may have this property, again contradicting.

Case 3. $deg_G\ v_{n-1} + deg_G\ v_n = 2n-4$ and $deg_H\ u_{n-1} + deg_H\ u_n = 4$.

Then $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-3\}$ *or* $\{n-2\}$.

Subcase 3.1. $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-3\}$.

Now $deg_G\ v_n = n-1$ and $deg_G\ v_{n-1} = n-3$. Then G has at most two endvertices and no isolated vertex. Since $deg_H\ u_{n-1} + deg_H\ u_n = 4$, it follows that $\{deg_H\ u_{n-1}, deg_H\ u_n\} \in \{\{0,4\},\{1,3\},\{2\}\}$. If $\{deg_H\ u_{n-1}, deg_H\ u_n\} \neq \{2\}$ or $|N(u_n) \cap N(u_{n-1})| \neq 2$, then H has a common card with no $(n-2)$-vertex. But each common card of G has at least one $(n-2)$-vertex (since $deg_G\ v_n = n-1$), giving a contradiction. We assume, therefore, that $\{deg_H\ u_{n-1}, deg_H\ u_n\} = \{2\}$ and $|N(u_n) \cap N(u_{n-1})| = 2$.

Let $N(u_n) \cap N(u_{n-1}) = \{u_r, u_s\}$. Then every common card of G (and hence H) has at least one $(n-2)$-vertex. On the otherhand, since $deg_H\ u_{n-1} = deg_H\ u_n = 2$, it follows that the only vertices in H that can have degree at least $n-2$ in H are u_r and u_s. As the result, we have $deg_H\ u_i = n-1$, $i \in \{r,s\}$. Thus, H has exactly two common cards (namely $H-u_r$ and $H-u_s$) with at least two endvertices adjacent to an $(n-2)$-vertex. But at most one or at least $n-4$ (≥ 7) common card of G may have such a property, giving a contradiction.

Subcase 3.2. $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-2\}$.

Then in G, there can be at most one vertex which is adjacent neither to v_n nor to v_{n-1} and that vertex alone can be isolated vertex of G. On the otherhand, G has at least $n-4$ vertices which are adjacent to v_n and v_{n-1}. Hence other than v_n and v_{n-1}, there can be at most two vertices in G which are nonadjacent to at least one among v_{n-1} and v_n, and these alone can be endvertices of G.

Thus, *G has at most one isolated vertex and at most two endvertices.* .. (5)

If $v_n \not\sim v_{n-1}$, then vertices v_n and v_{n-1} are adjacent to every other vertices of G and hence no common card of G has a vertex of degree 0 or 1. Since $deg_H\ u_{n-1} + deg_H\ u_n = 4$, it follows that at least one common card of H has an isolated vertex or endvertex, giving a contradiction. So we assume that $v_n \sim v_{n-1}$. Then there must be a vertex in G, say v_t which is nonadjacent to v_{n-1} or v_n. Consequently,

the common card $G - v_t$ has an $(n-2)$-vertex. ...... (6)

We also assume that $N(u_n) \cap N(u_{n-1}) \neq \phi$ (as otherwise, no common card of H has an $(n-2)$-vertex, contradicting (6)).

Then $\{deg_H\ u_n, deg_H\ u_{n-1}\} \in \{\{3,1\},\{2\}\}$ and so we have two subcases.

Subcase 3.2.1. $\{deg_H\ u_n, deg_H\ u_{n-1}\} = \{3,1\}$.

Since $N(u_n) \cap N(u_{n-1}) \neq \phi$, it follows that $u_n \not\sim u_{n-1}$. As $deg_H\ u_i \geq 2$ ($1 \leq i \leq n-2$), the vertex u_{n-1} is the only endvertex in H; the vertex u_s denotes the neighbour of u_{n-1}. The common card $H - u_s$ has exactly one isolated vertex. Each of the other $n-3$ common cards $H - u_i$ (and hence $G - v_i$), $i \in \{1, 2, ..., n-2\} - \{s\}$ has the following two properties.

(i) has no isolated vertex

(ii) has at least one endvertex. ...... (7)

As a result, G has an endvertex, say v'_s adjacent to v_s and so $v_i \not\sim v'_s$ for $i \in \{s, n-1, n\}$. Thus, $deg_G\ v'_s = 1$, $deg_G\ v_s \geq 3$ (because $v_s \sim v_{n-1} \neq v'_s$ and $v_s \sim v_n \neq v'_s$), and $deg_G\ v_i \geq 2$ for all other vertices v_i of G. Hence the common card $G - v'_s$ has no endvertex. Thus, at most $n-4$ common cards of G may have endvertices, contradicting (7).

Subcase 3.2.2. $\{deg_H\ u_{n-1}, deg_H\ u_n\} = \{2\}$.

If $u_n \sim u_{n-1}$, then each common card of H has at most one vertex of degree at least $n-3$. Since $deg_G\ v_n = deg_G\ v_{n-1} = n-2$, it follows that every common card of G has at least two vertices of degree at least $n-3$, giving a contradiction. We assume, therefore, that $u_n \not\sim u_{n-1}$. Then, since $N(u_n) \cap N(u_{n-1}) \neq \phi$, it follows that $|N(u_n) \cap N(u_{n-1})| = 1$ or 2. If $|N(u_n) \cap N(u_{n-1})| = 2$, then each common card of H has two endvertices adjacent to the same vertex. Consequently, each common card of G has two endvertices adjacent to the same vertex, which is a contradiction to $deg_G\ v_{n-1} = deg_G\ v_n = n-2$. Hence, let $|N(u_{n-1}) \cap N(u_n)| = 1$.

Since $deg_G\ v_n = deg_G\ v_{n-1} = n-2$ and $v_n \sim v_{n-1}$, we have (i) $|N(v_n) \cap N(v_{n-1})| = n-4$, or (ii) $|N(v_n) \cap N(v_{n-1})| = n-3$. If (i) holds, then there is a vertex, say v_α not adjacent to v_{n-1} and v_n, and hence the common card $G - v_\alpha$ (and hence $H - u_\alpha$) contains two $(n-2)$-vertices, contradicting the fact that $|N(u_{n-1}) \cap N(u_n)| = 1$. So we assume that (ii) holds; let v_α, v_β denote the vertices not adjacent to v_{n-1} and v_n, respectively. Then v_α and v_β are the only vertices that can be endvertices in a common card of G. Let $u_b \in N(u_n) \cap N(u_{n-1})$, $u_a \in N(u_n) \setminus N(u_{n-1})$, and $u_s \in N(u_{n-1}) \setminus N(u_n)$. Since $H - u_b$ ($\cong G - v_b$) has exactly two endvertices, it follows that the two endvertices in $G - v_b$ must be v_α and v_β. Consequently, the vertices u_a and u_s have degree at least $n-3$ in $H - u_b$ and hence they have degree at least $n-3$ in H. Hence exactly three common cards of H (namely $H - u_a$, $H - u_b$, $H - u_s$) have endvertices. But at least $n - 4$ (≥ 7) (this happens when v_α or v_β is an endvertex of G) or at most two (this happens when G has no endvertices) common cards of G may have endvertices, producing a contradiction.

Case 4. $deg_G\ v_{n-1} + deg_G\ v_n = 2n - 5$ and $deg_H\ u_{n-1} + deg_H\ u_n = 3$.

Now $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-4, n-1\}$ or $\{n-3, n-2\}$, and if the former holds, then every common card of G has at least one $(n-2)$-vertex. But H does not have this property, contradicting. Hence, let $deg_G\ v_n = n-2$ and $deg_G\ v_{n-1} = n-3$.

If $v_n \not\sim v_{n-1}$ or $|N(v_n) \cup N(v_{n-1})| = n$, then clearly no common card of G has a component isomorphic to K_1 or K_2. But H has a common card with a component isomorphic to K_1 or K_2 (Figure 3), giving a contradiction. Otherwise, $|N(v_{n-1}) \cup N(v_n)| = n - 1$. Let $v_s \notin N(v_n) \cup N(v_{n-1})$ and $v_t \in N(v_n) \setminus N(v_{n-1})$. Then the common card $G - v_s$ (and hence $H - u_s$) has the following two properties.

(i) has an $(n-2)$-vertex, namely v_n (and so $G - v_s$ has no isolated vertex)

(ii) has at most one endvertex (the vertex nonadjacent to v_{n-1}).

Also the common card $G - v_t$ contains two $(n-3)$-vertices which are adjacent, and no endvertex of $G - v_t$ is adjacent to these two $(n-3)$-vertices. Since $deg_H\ u_n + deg_H\ u_{n-1} = 3$, the graph H must be one of the four types shown in Figure 3. If H is Type 4, then no common card of H is isomorphic to $G - v_t$. Otherwise, no common card of H is isomorphic to $G - v_s$. This completes Case 4.

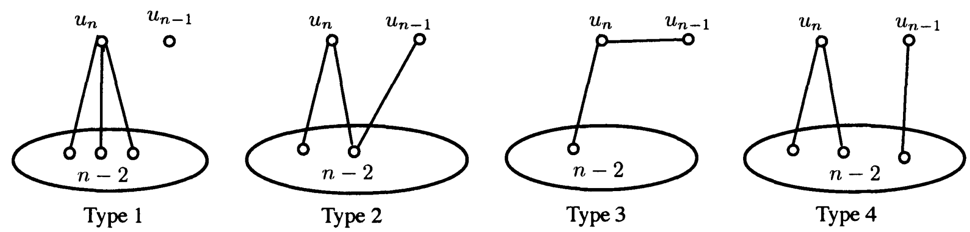

Figure 3: Structure of H in Case 4, Theorem 3.

Case 5. $deg_G\ v_{n-1} + deg_G\ v_n = 2n-6$ and $deg_H\ u_{n-1} + deg_H\ u_n = 2$.

Then $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-5, n-1\}, \{n-4, n-2\}$, or $\{n-3\}$. If $deg_G\ v_{n-1} = n-5$ and $deg_G\ v_n = n-1$, then every common card of G has at least one $(n-2)$-vertex. But H must have a common card with no $(n-2)$-vertex (Figure 4), contradicting. Next, let $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-2, n-4\}$. **If** $v_n \not\sim v_{n-1}$ or $|N(v_n) \cup N(v_{n-1})| = n$, then no common card of G has a component isomorphic to K_1 or K_2. But H must have a common card with a component isomorphic to K_1 or K_2 (Figure 4), again contradicting. **Otherwise** $|N(v_n) \cup N(v_{n-1})| = n-1$. Let $v_t \in N(v_n) \setminus N(v_{n-1})$. Then each of the $n-3$ common cards $G - v_i$ ($1 \leq i \leq n-2$ and $i \neq t$) of G has at most one isolated vertex, and the common card $G - v_t$ has an $(n-2)$-vertex (namely v_n) and has no isolated vertex. ...... (12)

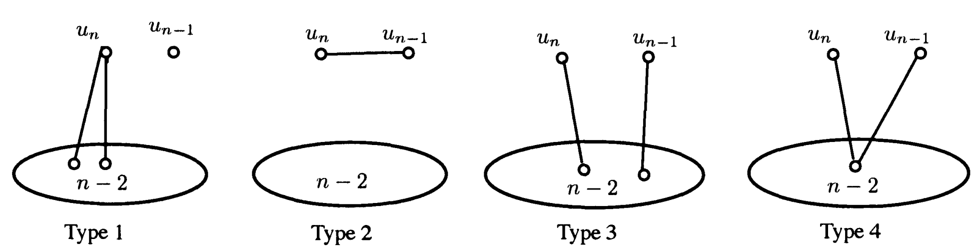

Figure 4: Structure of H in Case 5, Theorem 4.

Since $deg_H\ u_n + deg_H\ u_{n-1} = 2$, the graph H is one of the four types shown in Figure 4. If H is Type 4, then the common card $H - u_r$ has exactly two isolated vertices, contradicting (12). If H is not Type 4, then no common card of H has an $(n-2)$-vertex, again contradicting (12). The only remaining case that $\{deg_G\ v_n, deg_G\ v_{n-1}\} = \{n-3\}$ is proved to be impossible in Case 1 of Theorem 6 in [6] (where the proof dealt with E and F instead of G and H respectively and it does not used the vertices of degree at least $n-2$). This completes Case 5.

Case 6. $deg_G\ v_{n-1} + deg_G\ v_n = 2n - 7$ and $deg_H\ u_{n-1} + deg_H\ u_n = 1$.

Then $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-6\}, \{n-2, n-5\}$, or $\{n-3, n-4\}$. If $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-6\}$, then each common card of G has an $(n-2)$-vertex. But no common card of H has an $(n-2)$-vertex, giving a contradiction. If $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-2, n-5\}$, then G can have at most one isolated vertex and at most five endvertices. Since $deg_H\ u_i \geq 2$ for $i = 1$ to $n-2$, it follows that u_{n-1} is the only isolated vertex of H. Consequently, each common card of H and hence each common card of G has at least one isolated vertex. This is impossible when $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-2, n-5\}$. The only remaining case that $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-3, n-4\}$ is proved to be impossible in Case 2 of Theorem 6 in [6].

Case 7. $deg_G\ v_{n-1} + deg_G\ v_n = 2n - 8$ and $deg_H\ u_{n-1} + deg_H\ u_n = 0$.

Clearly $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-7\}, \{n-2, n-6\}, \{n-3, n-5\}$, or $\{n-4\}$. If $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-1, n-7\}$, then each common card of G has an $(n-2)$-vertex. But no common card of H has an $(n-2)$-vertex, contradicting. If $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-2, n-6\}$, then each common card of G has an $(n-3)$-vertex. But no common card of H has an $(n-3)$-vertex, again contradicting. The remaining two cases, namely $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-3, n-5\}$ and $\{deg_G\ v_{n-1}, deg_G\ v_n\} = \{n-4\}$ are proved to be impossible in Case 3 of Theorem 6 in [6]. Thus all the seven cases lead to a contradiction and complete the proof of Theorem 4.■

4 Conclusion

There are graph pairs G and H on eight vertices with six common cards such that $|e(G) - e(H)| = 1$. Rivshin and Radziszowski [7] reported that for $9 \leq n \leq 11$, there exists no pairs of non-isomorphic graphs with $n-2$ common cards. It appears that similar techniques can be used to find the absolute difference between the number of edges in pairs of graphs with $n-3$ common cards.

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